Joint dynamic delay and channel gain estimation BCRB for wireless communications

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***Abstract*—In this paper, we derive the analytical expression of the Bayesian Cramer-Rao Bound (BCRB) for the joint estimation of random time delay and channel gains. The BCRB is evaluated for the Rayleigh fading channel for both Data-Aided (DA) and Non-Data-Aided (NDA) processing modes. Simulation results make a comparison, over various channels, between several synchronization modes. In particular, we show that a better performance can be achieved by using DA and off-line estimation techniques, where all the received signal samples are used at any time of the estimation process.**

***Index Terms*—Time delay recovery, Channel Estimation, Bayesian Cramer-Rao Bound**

I. Introduction

Time synchronization is the first function performed by the demodulator. It is critical to the reception quality of the transmitted symbols. Indeed, it allows the receiver to minimize the InterSymbol Interference (ISI) by sampling the received signal at near-optimal sampling instants, which considerably impacts the overall system performance.

In a wireless multipath context, a conventional equalizer can restore, up to some theoretical limits, the offset between the samples when this offset is constant; however, this operation becomes very difficult at low Signal to Noise Ratio (SNR). Furthermore, when the time delay changes over consecutive samples, time synchronization becomes the only alternative to avoid inter-symbol interference. NDA timing recovery techniques seems to be interesting especially for low complexity wireless communication systems, in particular those deployed in IoT networks. This is why, it is necessary to further develop unsupervised time synchronization algorithms in order to decrease the network overhead, while trying to minimize the implementation complexity.

Synchronization error estimators are generally evaluated in terms of bias and Mean Square Error (MSE). However, it can be difficult to analytically derive these two parameters expressions. For instance, the authors in [1] propose semi-analytic expressions of the bias and variance of the estimators, as a function of time delay, for a Mueller&M$\ddot{u}$ller detector in a Code-Aided (CA) context, but the performance evaluation is only limited to the case of low SNR BPSK signals, and it is based on the assumption that the ISI can be approximated by a white Gaussian noise [2]. This illustrates that even for many classical cases, the performances of the estimators must be compared to some theoretical estimation error bounds.

Several theoretical bounds have been proposed in the literature, such as the Bhattacharyya bound [3], Chapman-Robbins bound [4], Barankin bound [5], [6], [7], [8], Abel bound [9] and the Cramer-Rao Bound (CRB) [10]. The CRB is more often used because it is easier to derive. Analytical expressions of the CRB have been obtained in [11] for a CA frequency and phase recovery for turbo-coded QAM-square signals. For the timing recovery problem, the CRB was evaluated for a DA scenario [12], NDA [13] and CA [14], [15] in the case of a constant delay for Gaussian fading channels; however most publications, such as [12], [13], [14], [15] do not consider the interesting but difficult case of a wireless link for which the amplitude is time-varying.

The standard CRB is poorly suited for the evaluation of random parameters or the joint estimation of random and deterministic parameters. The Modified CRB (MCRB) presented in [16], [17] is easier to obtain. Nevertheless, it fails significantly to approach the true CRB, especially at low SNR values. In this case, other theoretical bounds can be used, namely the Bayesian CRB and the Hybrid CRB (HCRB). Analytical expressions of the theoretical bounds were obtained for the phase and frequency estimation error. The BCRB was evaluated in [18] for a random time delay estimation in the case of a Gaussian channel. In [19], [20], [21], the authors evaluated the HCRB and the BCRB for the dynamical phase offset estimation for QAM signals for the CA, DA and NDA modes and theoretically showed the improvement provided by the use of a CA technique [22]. When the channel gain cannot be assumed constant over the received samples block, generally time synchronization is directly integrated into the channel estimation process. An evaluation of the HCRB has been presented in [23] for the joint channel gains and a constant time delay estimation which can be useful for VANET applications [24], [25]. However, explicit consideration of time synchronization is particularly important when the time offset cannot be assumed constant over the observation window, as in the case of a burst transmission over a fading channel [16]. In this paper, the BCRB is evaluated for the joint estimation of a random time delay and Rayleigh distributed fading channel gains. It is also used to prove the enhancement brought by the use of NDA timing recovery technique in an off-line context.

This paper is organized as follows: In section II, the system model is presented. In section III, the BCRB for the joint channel gains and time delay estimation is derived. Simulation results are provided in section IV. The last section concludes our work.

II. System Model

Let us consider the following transmitted signal $s(t)$:

$$s(t) = \sum_i a_i h(t - iT), \tag{1}$$

where a_i are linearly modulated transmitted symbols which are assumed to be statistically independent and equally likely, with normalized energy, $h(t)$ is the impulse response of the root Nyquist transmission filter and T is the symbol period.

At the receiver side, let us consider r_k the k^{th} matched filtered signal sample during the observation period T_0 which is given:

$$r_k = \alpha_k s_k(\tau_k) + n_k, \tag{2}$$

where α_k is the Rayleigh distributed channel gain, τ_k is the time delay, $s_k(\tau_k) = s(kT_s - \tau_k)$, T_s is the sampling period and n_k the k^{th} noise sample. The channel gain is assumed to be wide-sense stationary (WSS), a narrow-band zero-mean complex Gaussian process of variance σ_α^2 with the so-called Jakes' power spectrum of maximum Doppler frequency f_d [26]. Considering that f_d is very low with respect to the symbol rate $1/T$, thus the channel gains are supposed to be constant during a symbol period. We also assume that the delay τ_k follows a Wiener evolution model [27], [28] according to:

$$\tau_k = \tau_{k-1} + w_k, \tag{3}$$

where w_k is a stationary white Gaussian noise with zero mean and variance σ_w^2. Let us consider $\boldsymbol{r} = [r_1, \ldots, r_N]^T$, $\boldsymbol{a} = [a_1, \ldots, a_N]^T$, $\boldsymbol{\tau} = [\tau_1, \ldots, \tau_N]^T$ and $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_N]^T$.

The channel gain follows a Rayleigh distribution, therefore, $\boldsymbol{\alpha}$ is a complex Gaussian random variable of zero mean and covariance matrix $\boldsymbol{R_\alpha}$. The *a priori* information on $\boldsymbol{\alpha}$ is thus given by:

$$P(\boldsymbol{\alpha}) = \frac{1}{|\pi \boldsymbol{R_\alpha}|} \exp\left(-\boldsymbol{\alpha}^H \boldsymbol{R_\alpha}^{-1} \boldsymbol{\alpha}\right). \tag{4}$$

The element of the n^{th} row m^{th} column of the covariance matrix is given by [29]:

$$[\boldsymbol{R_\alpha}]_{n,m} = \sigma_\alpha^2 J_0\left(2\pi f_d T(n-m)\right), \tag{5}$$

where $J_0(.)$ is the Bessel function of the first kind.

III. Bayesian Cramer-Rao Bound on joint channel gains and time delay estimation

Let us consider the random vector $\boldsymbol{\mu} = [\boldsymbol{\alpha}, \boldsymbol{\tau}]$. Let $P(\boldsymbol{\mu})$ be the *a priori* probability of the vector $\boldsymbol{\mu}$. The BCRB on the estimation of the random vector $\boldsymbol{\mu}$ is therefore obtained by inverting the following Bayesian Information Matrix (BIM) [30]:

$$\boldsymbol{B_\mu} = E_{\boldsymbol{\mu}}[\boldsymbol{F(\mu)}] + E_{\boldsymbol{\mu}}[-\Delta_{\boldsymbol{\mu}}^{\boldsymbol{\mu}} \log\left(P(\boldsymbol{\mu})\right)], \tag{6}$$

where $E_{\boldsymbol{\mu}}$ and $\Delta_{\boldsymbol{\mu}}^{\boldsymbol{\mu}}$ are respectively the expectation and the Laplacian operator with respect to the vector $\boldsymbol{\mu}$; $\boldsymbol{F(\mu)}$ is similar to the conventional Fisher Information Matrix (FIM) [10]:

$$\boldsymbol{F(\mu)} = E_{\mathbf{r}|\boldsymbol{\mu}}[-\Delta_{\boldsymbol{\mu}}^{\boldsymbol{\mu}} \log\left(P(\mathbf{r}|\boldsymbol{\mu})\right)]. \tag{7}$$

The first term in (6) is the expectation with respect to $\boldsymbol{\mu}$ of the information provided by the observation vector $\boldsymbol{r}$. The second term depends on the *a priori* information on $\boldsymbol{\mu}$. The diagonal elements of the inverse of $\boldsymbol{B_\mu}$ represent the BCRB expression for the estimation of $\boldsymbol{\mu}$. Thus, the BCRB consists of four blocks of sub-matrices:

$$\boldsymbol{BCRB(\mu)} = \begin{pmatrix} \boldsymbol{BCRB}^{11}(\boldsymbol{\alpha}) & \boldsymbol{BCRB}^{12}(\boldsymbol{\alpha},\boldsymbol{\tau}) \\ \boldsymbol{BCRB}^{21}(\boldsymbol{\alpha},\boldsymbol{\tau}) & \boldsymbol{BCRB}^{22}(\boldsymbol{\tau}) \end{pmatrix} \tag{8}$$

The first term in (6) is the expectation with respect to $\boldsymbol{\mu}$ of the FIM $\boldsymbol{F(\mu)}$ given by the following sub-matrices:

$$\boldsymbol{F(\mu)} = \begin{pmatrix} \boldsymbol{F}^{11}(\boldsymbol{\alpha}) & \boldsymbol{F}^{12}(\boldsymbol{\alpha},\boldsymbol{\tau}) \\ \boldsymbol{F}^{21}(\boldsymbol{\alpha},\boldsymbol{\tau}) & \boldsymbol{F}^{22}(\boldsymbol{\tau}) \end{pmatrix}, \tag{9}$$

where:

$$\boldsymbol{F}^{11}(\boldsymbol{\alpha}) = E_{\mathbf{r}|\boldsymbol{\alpha},\tau}\left[-\triangle_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}} \log\left(P(\mathbf{r}|\boldsymbol{\alpha},\tau)\right)\right], \tag{10}$$

$$\boldsymbol{F}^{12}(\boldsymbol{\alpha},\boldsymbol{\tau}) = E_{\mathbf{r}|\boldsymbol{\alpha},\boldsymbol{\tau}}\left[-\triangle_{\boldsymbol{\alpha}}^{\boldsymbol{\tau}} \log\left(P(\mathbf{r}|\boldsymbol{\alpha},\boldsymbol{\tau})\right)\right], \tag{11}$$

$$\boldsymbol{F}^{21}(\boldsymbol{\alpha},\boldsymbol{\tau}) = E_{\mathbf{r}|\boldsymbol{\alpha},\boldsymbol{\tau}}\left[-\triangle_{\boldsymbol{\tau}}^{\boldsymbol{\alpha}} \log\left(P(\mathbf{r}|\boldsymbol{\alpha},\boldsymbol{\tau})\right)\right], \tag{12}$$

$$\mathrm{F}^{22}(\boldsymbol{\tau}) = E_{\mathbf{r}|\boldsymbol{\alpha},\boldsymbol{\tau}}\left[-\triangle_{\boldsymbol{\tau}}^{\boldsymbol{\tau}} \log\left(P(\mathbf{r}|\boldsymbol{\alpha},\boldsymbol{\tau})\right)\right]. \tag{13}$$

Given that $\boldsymbol{\alpha}$ and $\boldsymbol{\tau}$ are two independent random variables, thus $\boldsymbol{F}^{12}(\boldsymbol{\alpha},\boldsymbol{\tau}) = \boldsymbol{F}^{21}(\boldsymbol{\alpha},\boldsymbol{\tau}) = \boldsymbol{0_{N,N}}$ and therefore:

$$E_{\boldsymbol{\mu}}[\boldsymbol{F(\mu)}] = \begin{pmatrix} E_{\boldsymbol{\alpha}}\left[\boldsymbol{F}^{11}(\boldsymbol{\alpha})\right] & \boldsymbol{0_{N,N}} \\ \boldsymbol{0_{N,N}} & E_{\boldsymbol{\tau}}\left[\boldsymbol{F}^{22}(\boldsymbol{\tau})\right] \end{pmatrix}. \tag{14}$$

In the following, we distinguish two types of BCRB, namely, the BCRB for the on-line estimation mode where only the current and previous observations are used for the estimation of the current time delay and the BCRB for the off-line estimation mode where the entire observation block is used for the estimation of the current time delay. The off-line BCRB expression is given at each sample k of a block of N observations by the k^{th} diagonal element of the inverse of the BIM (6). However, the on-line BCRB is given by the last diagonal element N of the inverse of the BIM for a block of N observations.

A. Off-line Bayesian Cramer-Rao Bound

Let us start with the derivation of the second term in (6). Given that $\boldsymbol{\alpha}$ and $\boldsymbol{\tau}$ are two independent parameters, we obtain:

$$E_{\boldsymbol{\mu}}\left[-\triangle_{\boldsymbol{\mu}}^{\boldsymbol{\mu}} \log\left(P(\boldsymbol{\mu})\right)\right] = \begin{pmatrix} E_{\boldsymbol{\alpha}}\left[-\triangle_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}} \log\left(P(\boldsymbol{\alpha})\right)\right] & \boldsymbol{0_{N,N}} \\ \boldsymbol{0_{N,N}} & \left[-\triangle_{\boldsymbol{\tau}}^{\boldsymbol{\tau}} \log\left(P(\boldsymbol{\tau})\right)\right] \end{pmatrix} \tag{15}$$

From (4), we have:

$$E_{\boldsymbol{\alpha}}\left[-\triangle_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}} \log\left(P(\boldsymbol{\alpha})\right)\right] = \boldsymbol{R_\alpha}^{-1}. \tag{16}$$

Based on (3), the second term of the diagonal sub-matrices in (15) is given by:

$$E_{\boldsymbol{\tau}}[\triangle_{\boldsymbol{\tau}}^{\boldsymbol{\tau}} \log\left(P(\boldsymbol{\tau})\right)] = \begin{pmatrix} \frac{1}{\sigma_w^2}-G & -\frac{1}{\sigma_w^2} & 0 & \cdots & & 0 \\ -\frac{1}{\sigma_w^2} & \frac{2}{\sigma_w^2} & -\frac{1}{\sigma_w^2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & -\frac{1}{\sigma_w^2} & \frac{2}{\sigma_w^2} & -\frac{1}{\sigma_w^2} \\ 0 & \cdots & 0 & 0 & -\frac{1}{\sigma_w^2} & \frac{1}{\sigma_w^2} \end{pmatrix} \quad (17)$$

where $G = E_{\tau_1}[\frac{\partial^2 \log\left(P(\tau_1)\right)}{\partial \tau_1^2}]$. In practice, the receiver has no *a priori* information on τ_1, thus $G=0$.

Let us move on now to the derivation of the first term of (6) given by the expectation with respect to $\boldsymbol{\mu}$ of (9).

According to (13), the expression of $\mathrm{F}^{22}(\boldsymbol{\tau})$ depends on $P(\mathbf{r}|\boldsymbol{\tau},\boldsymbol{\alpha})$.

In order to derive the expression of $P(\mathbf{r}|\boldsymbol{\tau},\boldsymbol{\alpha})$, we need to evaluate the expression of $P(\mathbf{r}|\boldsymbol{u},\boldsymbol{\alpha})$ for an estimated time delay vector $\boldsymbol{u}$ which is equal to the real time delay $\boldsymbol{\tau}$. For a given $\mathbf{u}$, $\boldsymbol{\alpha}$ and $\mathbf{a}$, we have that:

$$P(\mathbf{r}|\mathbf{u},\mathbf{a},\boldsymbol{\alpha}) \quad = \quad \prod_{i=1}^{N} P(r_i|u_i,\mathbf{a},\alpha_i). \quad (18)$$

The expression of (18) can be easily obtained based on the likelihood probability derivation that we presented in [23] and in [18], we obtain:

$$P(\mathbf{r}|\mathbf{u},\mathbf{a},\boldsymbol{\alpha}) = \left(\frac{C}{2\pi\sigma_n^2}\right)^N \exp\left(\sum_{i=1}^{N}\left(\frac{\Re\left\{\alpha_i^* a_i^* x_i(u_i)\right\}}{\sigma_n^2} - \frac{|\alpha_i a_i|^2}{2\sigma_n^2}\right)\right), \quad (19)$$

where C is a constant term with respect to $\boldsymbol{u}$ and,

$$x_i(u_i) \quad = \quad y_i(u_i) + v_i(u_i), \quad (20)$$

$$y_i(u_i) \quad = \quad \sum_l \alpha_l a_l g\left((i-l)T - (\tau_i - u_i)\right), \quad (21)$$

$$v_i(u_i) \quad = \quad \int_{T_0} h(t - iT - u_i) n(t) dt, \quad (22)$$

$$g(t) \quad = \quad h(t) \otimes h^*(-t). \quad (23)$$

By choosing $h(.)$ as a square root raised cosine filter then $g(.)$ given by (23) is a Nyquist filter.

Based on (19), we can deduce that $\boldsymbol{F^{22}(\tau)}$ is a diagonal matrix. Thus, the first term of (6) can be written as:

$$E_{\boldsymbol{\tau}}[\boldsymbol{F(\tau)}] = \boldsymbol{D}, \quad (24)$$

where $\boldsymbol{D}$ is a diagonal matrix and the diagonal k^{th} element is given by:

$$[\boldsymbol{D}]_{k,k} \quad = \quad E\left[-\frac{\partial^2 \log\left(P(r_k|\tau_k,\alpha_k)\right)}{\partial \tau_k^2}\right]. \quad (25)$$

Since the expression of $[\boldsymbol{D}]_{k,k}$ is independent of the index k, the matrix $\boldsymbol{D}$ can rewritten as. The diagonal elements of the matrix $\boldsymbol{D}$ are then equal and therefore it can be written as:

$$\boldsymbol{D} = J_D \boldsymbol{I_N} \quad (26)$$

where $\boldsymbol{I_N}$ is the identity matrix $N \times N$.

On the other hand, according to the expression of $P(\mathbf{r}|\boldsymbol{\tau},\boldsymbol{\alpha})$ given by (19):

$$\boldsymbol{F(\alpha)^{11}} = \mathrm{diag}(J_1, \cdots, J_N), \quad (27)$$

where $\mathrm{diag}(\boldsymbol{v})$ refers to a diagonal matrix where the diagonal elements are given by the vector $\boldsymbol{v}$ and J_k is given by:

$$J_k = \frac{|a_k|^2}{\sigma_n^2}. \quad (28)$$

As a result, we get:

$$E_{\boldsymbol{\alpha}}\left[\boldsymbol{F(\alpha)^{11}}\right] = diag(J_1, \cdots, J_N). \quad (29)$$

Consequently, based on (6), we obtain:

$$\boldsymbol{B_\mu} = \begin{pmatrix} E_{\boldsymbol{\alpha}}\left(\boldsymbol{F}^{11}(\boldsymbol{\alpha})\right) + \boldsymbol{R_\alpha}^{-1} & \boldsymbol{0_{N,N}} \\ \boldsymbol{0_{N,N}} & \boldsymbol{G_N} \end{pmatrix}, \quad (30)$$

with:

$$\boldsymbol{G_N} = \beta \begin{pmatrix} A+1 & 1 & 0 & \cdots & & 0 \\ 1 & A & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & 1 & A & 1 \\ 0 & \cdots & 0 & 0 & 1 & A+1 \end{pmatrix}, \quad (31)$$

where $A = -\sigma_w^2 J_D - 2$ and $\beta = -\frac{1}{\sigma_w^2}$.

The matrix $\boldsymbol{B_\mu}$ is a block diagonal matrix. Using the block matrices inversion formula [10], its inverse is given by:

$$\boldsymbol{B_\mu}^{-1} = \begin{pmatrix} \left(E_{\boldsymbol{\alpha}}\left[\boldsymbol{F}^{11}(\boldsymbol{\alpha})\right] + \boldsymbol{R_\alpha}^{-1}\right)^{-1} & \boldsymbol{0_{N,N}} \\ \boldsymbol{0_{N,N}} & \boldsymbol{G_N}^{-1} \end{pmatrix}. \quad (32)$$

The expression of $[\boldsymbol{G_N}^{-1}]_{k,k}$, which leads directly to the expression of the off-line BCRB given by [18]:

$$[\boldsymbol{G_N}^{-1}]_{k,k} \quad = \quad \frac{1}{|G_N|}\Bigg[\rho_1^2(\beta+\nu_1)^2\nu_1^{N-3} + \rho_2^2(\beta+\nu_2)^2\nu_2^{N-3} - \frac{\beta^2}{A-2}(\nu_1^{k-2}\nu_2^{N-k-1} + \nu_1^{N-k-1}\nu_2^{k-2})\Bigg], \quad (33)$$

where $|\boldsymbol{G_N}|$ is the determinant of $\boldsymbol{G_N}$ given by:

$$|\boldsymbol{G_N}| = (A+2)\beta\left(\rho_1\nu_1^{N-1} + \rho_2\nu_2^{N-1}\right), \quad (34)$$

and for $m = 1, 2$:

$$\nu_m \quad = \quad \frac{1}{\sigma_w^2} + \frac{J_D}{2}\left(1 + (-1)^m \times \sqrt{1 + \frac{4}{J_D\sigma_w^2}}\right), \quad (35)$$

$$\rho_m \quad = \quad \frac{\sqrt{1+\frac{4}{\sigma_w^2 J_D}} + (-1)^m \times (1 + \frac{2}{\sigma_w^2 J_D})}{2\sqrt{1+\frac{4}{\sigma_w^2 J_D}}}. \quad (36)$$

We hereafter present the derivation of the J_D expression in the DA context, where the transmitted symbols are known by the receiver.

1) J_D derivation in the DA context: In the DA context and based on (19), we obtain:

$$\frac{\partial^2 \log\left(P(r_k|\tau_k,\alpha_k)\right)}{\partial\tau_k^2} = -\frac{|\alpha_k|^2|a_k|^2}{\sigma_n^2}\ddot{g}(0) \approx -\frac{\rho}{N}\ddot{g}(0), \tag{37}$$

with $\rho = \frac{\sum_{k=1}^N |\alpha_k|^2|a_k|^2}{\sigma_n^2}$ is the SNR.

Thus:

$$E\left[-\frac{\partial^2 \log\left(P(r_k|\tau_k,\alpha_k)\right)}{\partial\tau_k^2}\right] = \frac{\bar{\rho}}{N}\ddot{g}(0), \tag{38}$$

where $\bar{\rho}$ is the mean SNR per symbol.

*2) J_D **derivation in the NDA context**:* In the NDA context, the expression of J_D is given by:

$$J_D = E\left[-\frac{\partial^2 \log\left(P(r_k|\tau_k)\right)}{\partial\tau_k^2}\right]. \tag{39}$$

In this paragraph, we provide the expression of the J_D components in the NDA mode for square-QAM modulated signals. The authors in [13] have computed the likelihood function expression for a block of N received square-QAM samples through an AWGN channel. Similarly, we can derive the expressions of the second derivative of the likelihood probability in (39) averaged only with respect to the observation noise for BPSK and Square-QAM modulated signals L_0 and L_p respectively where:

$$L_0 = -4\rho\left[\left(\frac{e^{-\rho}\beta(\rho)}{\sqrt{2\pi}}\right)\left(\rho\sum_{n=1}^N \dot{g}^2(nT) - \frac{\ddot{g}(0)}{2}\right) - \frac{\ddot{g}(0)}{2}\right], \tag{40}$$

$$\beta(\rho) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{x^2}{2}}}{\cosh\left(\sqrt{2\rho}x\right)}dx, \tag{41}$$

and

$$L_p = \frac{2\left(2\rho^2\sum_{n=1}^N \dot{g}^2(nT) - 2\rho\ddot{g}(0)\right)}{\sqrt{M\pi}}\int_{-\infty}^{+\infty}\frac{g_\rho(x)^2}{G_\rho(x)}e^{-\frac{x^2}{2}}dx \tag{42}$$

where:

$$g_\rho(x) = \sum_{k=1}^{2^{p-1}} \exp\left(-\rho(2k-1)^2 d_p^2\right)\sqrt(2)(2k-1)d_p \times \sinh\left(\sqrt(2\rho)(2k-1)d_p x\right), \tag{43}$$

and:

$$G_\rho(x) = \sum_{k=1}^{2^{p-1}} \exp\left(-\rho(2k-1)^2 d_p^2\right)\cosh\left(\sqrt(2\rho)(2k-1)d_p x\right), \tag{44}$$

$M = 2^{2p}$ is the constellation size and d_p is the inter-symbol distance which has the following expression for symbols with normalized energy:

$$d_p = \frac{2^{p-1}}{\sqrt{2^p\sum_{k=1}^{2^{p-1}}(2k-1)^2}}. \tag{45}$$

For Rayleigh distributed channel gains, the SNR ρ follows an exponential distribution. Its probability distribution function ρ is given by:

$$P(\rho) = \frac{1}{\bar{\rho}}\exp\left(-\frac{\rho}{\bar{\rho}}\right), \text{ for } \rho \geq 0. \tag{46}$$

In order to obtain the final expression of J_D, we need to average the analytical expressions of L_p for square-QAM modulated samples over the Probability Dstribution Function (PDF) of ρ for $\rho \geq 0$. Since it is hard to obtain analytically a closed form expression of the integrals given by (42) and average the obtained result with respect the SNR, a numerical integration can be used. We note that the integral in (42) decreases rapidly with respect to x. Thus, the integrand function can be approximated by a finite Riemann integration over an interval $[-C,+C]$ instead of $]-\infty,+\infty[$ with an integration step δ. The same technique can be used to average the obtained result with respect to the SNR on an interval $[0,C]$ instead of $[0,+\infty[$.

B. On-line Bayesian Cramer-Rao Bound

In the on-line mode. Only past and current observations are available to the receiver for the estimation of the time delay τ_k. The on-line BCRB expression can be given by the value of the sequential bound [31] at the index k, C_k. According to [18], C_k is equal to the inverse of the Bayesian information matrix at the index (k,k). In other words, the value of the on-line bound at the sample k is the same as that of the off-line bound at the end of a samples block of size k. So, we get:

$$C_k = [\boldsymbol{B_\mu}^{-1}]_{k,k}. \tag{47}$$

it is worth mentioning that, at the last sample of the observation block, the same amount of information is provided with both the on-line and off-line techniques.

From (8) and (30), we note that $\boldsymbol{BCRB}^{11}(\boldsymbol{\alpha})$ has the same expression as the one obtained for a constant time delay in [23]. For a time varying delay, only the expression of $\boldsymbol{BCRB}^{22}(\boldsymbol{\tau})$, the BCRB with respect to $\boldsymbol{\tau}$, changes. Therefore, in the following section, simulation results only concern the case of a random time delay.

IV. Simulation Results

In this paragraph, some simulation results are provided to evaluate the proposed expression of the BCRB for BPSK and 16QAM constellations. The evaluation of this bound gives an indication about the performance limit that can be achieved by a time synchronizer in the case of a Rayleigh fading channel. The global transmission filter is a raised cosine filter with roll-off factor equal to 0.3. The time delay varies according to the Wiener model with standard deviation $\sigma_w = 10^{-3}$. Figure 1 depicts the on-line and the off-line bounds for the joint estimation of the time delay and the channel gains at a SNR $= 5$ dB in the case a Gaussian channel and a Rayleigh channel with a Doppler shift characterized by $f_dT = 0.001$. The observation block length N is equal to 90.

It can be observed that the Bayesian bounds in the case of the Rayleigh fading channel have the same shape as those obtained for a Gaussian channel but translated upwards. As expected, an improvement in the estimation is seen with the off-line technique. This is due to the fact that an off-line algorithm explores all the observed samples for the estimation of the time delay at time instant k, whereas for the on-line process, the estimator only uses the observed value at time instant $k-1$ to update the estimated value at time instant k. We also note that the BCRB in the DA estimation mode is lower than that obtained for the NDA estimation mode. This enhancement is due to the use of the exact value of the transmitted symbols in the DA estimation process instead of an estimated value as in the NDA case. It is remarkable to note that an off-line NDA technique can outperform a DA on-line technique especially in the case of a Rayleigh fading channel.

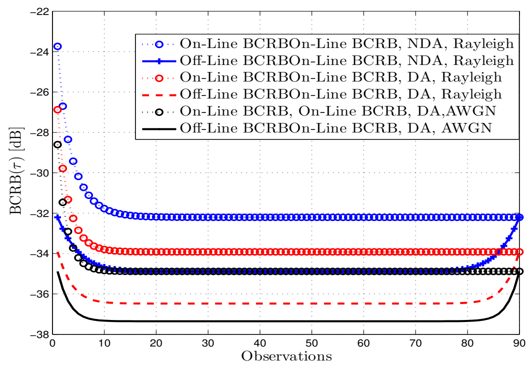

Fig. 1. $\boldsymbol{BCRB_{\tau}}$ versus Observation block sample, BPSK modulation, $N=90$, $f_dT=0.001$, SNR=5 dB.

In Figure 2, we evaluate the BCRB on $\boldsymbol{\tau}$ for various observation block lengths in the case of a Rayleigh channel characterized by $f_dT=0.001$ which means that the channel varies relatively slowly within the observation block. In the same figure, we also provide the mean square error (MSE) evaluation of two timing error detectors (TED): the Gardner [32] (GD) as a NDA scheme and the Zero Crossing Detector (ZCD) [32] as a DA scheme. The TEDs are evaluated for a Wireless Personal Area Network (WPAN) frame according to the standard IEEE 802.15.4 [33]. For that purpose, we consider a Physical Protocol Data Unit (PPDU) composed of a 40-samples BPSK modulated preamble, followed by a BPSK modulated PHY header and a Physical Service Data Unit (PSDU). The preamble sequence is known by the receiver, whereas, the PHY header and the PSDU are unknown sequences. The block of symbols is passed through a square root raised cosine transmission filter with a roll-off factor equal to 0.3. Then the signal is transmitted through a Rayleigh channel which introduces a random random time delay. At the receiver, the signal is matched filtered. For the preamble sequence, the timing recovery algorithm operates in the DA mode. For the next received data sequence, the TED switches to the NDA mode. The MSE is evaluated for both on-line and an off-line context. It is clearly shown that the MSE of the TED decreases when increasing the observation block size. It is worth mentioning that a supplemental performance enhancement can be achieved by an off-line technique when increasing the observation block length N, such that an even shorter DA preamble could be imagined so as to save both energy and bandwidth.

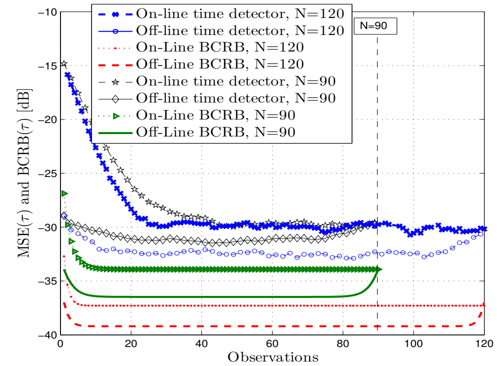

Fig. 2. $\boldsymbol{MSE_{\tau}}$ and data-aided $\boldsymbol{BCRB_{\tau}}$ versus Observation block sample, BPSK modulation, $f_dT=0.001$, SNR=5 dB.

These results are confirmed in Figure 3 which shows the value of the on-line and the off-line BCRB on $\boldsymbol{\tau}$ in the center of the observation block for different SNR values. This figure highlights the degradation due to the Rayleigh channel when compared to the AWGN channel. One also notes a degradation of the performances by increasing the product f_dT in the case of fast fading channels. It is also worth noting that the on-line and the off-line modes lead to similar theoretical performance at low SNRs and high SNRs, over the Rayleigh fading channel. Indeed, at low SNR, the use of an off-line algorithm does not improve the system performance compared to an on-line algorithm due to the severe noise degradation. At high SNR, the *a priori* information used by the off-line technique no longer impacts the estimation process since the received signal is reliable enough. However at medium SNRs, where real systems are constrained to operate, there is a clear improvement allowed by the off-line approach

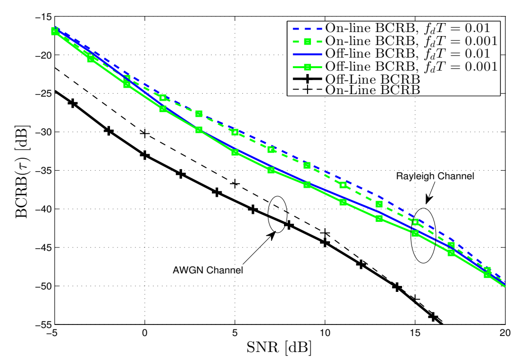

Fig. 3. NDA $\boldsymbol{BCRB_{\tau}}$ in terms of SNR, BPSK modulation, $N=40$.

In Figure 4, we evaluate the on-line and the off-line BCRB on $\boldsymbol{\tau}$ in the center of the observation block when $N=100$ for various SNR values for both 16QAM and BPSK constellations in the case of a Rayleigh fading channel. As expected, a lower BCRB is observed with the off-line technique and a larger block size (with respect to the results of the previous figure) and a higher BCRB is obtained with larger order constellations in both DA and NDA contexts.

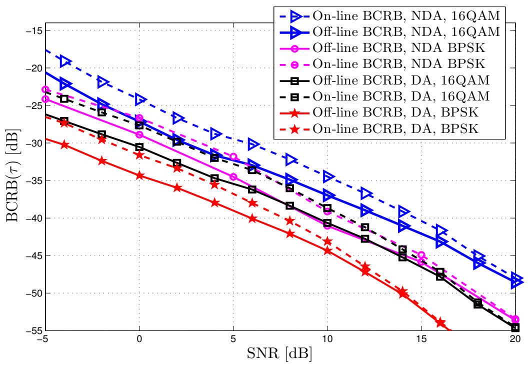

Fig. 4. $\boldsymbol{BCRB_\tau}$ in terms of SNR, $f_d T = 0.001$, $N = 100$.

V. Conclusion

In this paper, we evaluated the analytical expression of the BCRB for the joint time-varying delay and channel gain in the case of Rayleigh fading channels for DA and NDA modes. The simulation results show that we can achieve better estimation of the channel gains using off-line estimation techniques. In particular, we showed that off-line estimation techniques in the NDA mode theoretically outperform on-line estimation techniques operating in the DA mode over various channel conditions. Therefore, unsupervised off-line estimators are of a considerable interest, especially for narrowband wireless communication systems with high power efficiency requirements.

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