Towards the Complexity of the Widest Path Problem in Hybrid Multi-Channel WMNs

Martin Backhaus and Guenter Schaefer
Technische Universität Ilmenau, Germany
[martin.backhaus, guenter.schaefer][at]tu-ilmenau.de

***Abstract*—Finding a widest path to transmit the maximum possible data rate is well-known in the field of computer science. For computational complexity, the network type has a significant impact: It is relatively easy to solve for simple graphs, whereas it is NP-complete for wireless networks based on slotted time models. These models neither facilitate the problem of finding widest paths, nor are they best suited to reflect realistic networks based on IEEE 802.11. Therefore, this paper studies the widest path problem for hybrid multi-channel Wireless Mesh Networks (WMNs) without slotted time. In our model, wireless data rates are equally shared among edges within interference range. We prove NP-completeness of the widest path problem (even for a simpler model), but heuristics already demonstrated good results in practical settings.**

***Index Terms*—Wireless Mesh Networks, Hybrid Topologies, Widest Path Problem, NP-Completeness**

I. Introduction

The widest path problem is and has always been an integral part of computer science. It depicts one the most fundamental problems of routing in communication networks, when the focus is to maximize achievable data rates on paths through the network.

Solving the widest path problem is unproblematic in conventional networks modeled as simple graphs [1], but the computational complexity highly depends on the network type. When examining overlay networks for example, the problem of finding a widest path becomes NP-complete [2] due to edges that cannot be used independently from each other. Naturally, complexity increases also when considering wireless networks, as they are prone to the same dependence issue when examining certain edges. Additionally, multi-channel wireless networks are often modeled using slotted time [3], [4], which tremendously contributes to complexity. The issue with slotted time models is that they require meticulous planning and scheduling of transmissions on a network-wide scale to be put into practice, rendering these models unrealistic for distributed Wireless Mesh Networks (WMNs) based on IEEE 802.11, where nodes usually operate independently from each other (following local rules only). Finding a widest path in a wireless network based on slotted time is also NP-complete, as it requires an optimal graph coloring [3].

This is why this paper studies the widest path problem in multi-hop multi-channel hybrid WMNs based on a much simpler model, where achievable data rates depend on the set of wireless edges within an interference range. We also incorporate wired edges in the model, which are likely to occur in practical WMN instances, e.g., if nodes are in close proximity to each other. Therefore, hybrid topologies depict a highly relevant use case in practical scenarios. Fig.1 shows the scenario of a widest path from a source node s to a destination node t in a hybrid WMN. Although we will show that the widest path problem is still NP-complete using our simpler model, we also propose a very promising heuristic algorithm. Unfortunately, the heuristic's approximation error cannot be bounded, but it already proved to be very effective in real-world-like topologies [5].

This short paper provides the following contributions:

1) A proof for NP-completeness of the widest path problem for hybrid multi-channel WMNs.
2) A heuristic based on Dijkstra's algorithm.

The remainder of this paper is organized as follows: First, Sec. II briefly summarizes the state of the art for widest path complexity. Sec. III introduces definitions and the formal problem statement. The NP-completeness proof as well as the heuristic algorithm reside in Sec. IV. The paper concludes with directions for future research.

II. Related Work

The body of related work on the topic of widest paths is extraordinarily large, but the focus often differs. First, there are many different routing approaches and distributed protocol designs for widest paths in wireless networks (e.g. [6], [7]), but this paper focuses solely on the theoretical aspects of the problem, i.e., computational

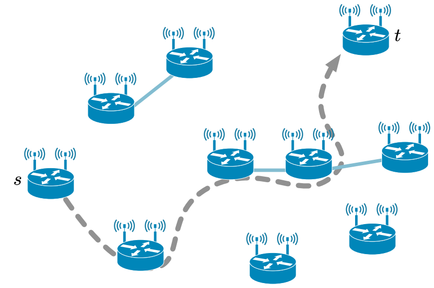

Fig. 1. Widest path in a hybrid WMN from a source s to a target node t. Network is comprised of Ethernet and wireless connections.

complexity. Furthermore, we do not cover multiple additive or independent metrics (e.g. [8]), but focus only on a singular data rate metric, only.

When it comes to path search in network graphs, Dijkstra's algorithm needs to be mentioned as one of the fundamental algorithms [9], but it is designed to find paths which are minimal with respect to a cost metric. With a few adaptations to metric handling (not to the algorithm itself), Dijkstra's algorithm is able to find widest paths in single-layer networks modeled as simple graphs [1]. Therefore, finding widest paths in simple graphs requires polynomial time. There has been a study on widest paths in overlay networks, discovering NP-completeness of the problem in this type of networks [2]. As already mentioned in the introduction, widest path approaches for wireless communication usually cover slotted time models and render the widest path problem NP-complete as it requires a solution of graph coloring problems to build global transmission schedules [3], [4].

In this paper, we employ a simpler, yet more realistic model to change the perspective on the problem. Instead of time slots (which lead to global schedules) we simply assume that edges within an interference range need to share their achievable data rate. With this model, we can show that widest path is still NP-complete, but develop a very promising heuristic to the problem. To the best of our knowledge, this paper is the first to conduct fundamental examinations of the complexity of finding a widest path in hybrid multi-channel WMNs.

III. Definitions and Problem Statement

As for our general notation[1], we distinguish between wired components of the network using a *bar* above an identifier (e.g. $\bar{G}$) and a *tilde* (e.g. $\tilde{G}$) to indicate wireless components. An instance of a hybrid multi-channel WMN an be formalized as a tuple $(\bar{G} = (\bar{V}, \bar{E}), \tilde{G} = (\tilde{V}, \tilde{E}, \mathcal{C}), \mathcal{P})$. $\bar{G}$ is the wired and $\tilde{G}$ the wireless graph. $\mathcal{P} : \bar{V} \cup \tilde{V} \mapsto \mathbb{R}^2$ assigns each node a two-dimensional Euclidean coordinate. The function $\mathcal{C}$ assigns each wireless node a set of channels to which its wireless interfaces are tuned to. Please note that $\tilde{E}$ highly depends on $\mathcal{C}$ and $\mathcal{P}$: An edge $(u, v) \in \tilde{E}$ if $\mathcal{C}(u) \cap \mathcal{C}(v) \neq \emptyset$ and their Euclidean distance (derived from $\mathcal{P}$) is less than a fixed *communication range*. For simplicity, we assume a fixed data rate for any edge in $\tilde{E}$, which is $\tilde{r}$. More importantly, all edges within *interference range* are said to share their achievable data rate equally. Again for simplicity, we assume the interference range to be twice as large as the communication range. Concerning wired parts of the network, an edge in $\bar{E}$ is said to have a fixed rate of $\bar{r}$. We assume $\tilde{r} \ll \bar{r}$.

Based on this model, the problem of finding a widest path is to obtain a path fulfilling the following constraint: The minimum data rate of all edges included in the path is maximal among all paths in the network. Clearly, a path's achievable data rate is bounded by the slowest rate on all links. But unlike wired edges, wireless edges share their rate among all other edges within interference range – possibly reducing the resulting path's data rate.

[1]This notation is taken from a more complex model presented in [5].

IV. Complexity of Finding a Widest Path in Hybrid Multi-Channel WMNs

This section examines the problem of Hybrid Wireless Multi-Channel Widest Path (HWMCWP) based on definitions and the model of the previous section. The much easier model will not render the problem simpler, but represents WMNs based IEEE 802.11 more realistically, as no global transmission schedules are required. The problem of finding a widest path is still NP-complete, but as we will see, the proof's construction generates topologies not remotely resembling realistic scenarios, indicating that simple heuristic solutions could be promising – especially when compared to graph coloring problems, which arise from a slotted time model. A thorough evaluation of the impact of HWMCWP's NP-completeness in practically relevant settings would require optimal results, e.g., using integer linear programming formulations or combinatorial approaches, which is beyond the scope of this short paper.

A. NP-Completeness

The proof's idea is similar to the one in [2], where overlay networks are covered. First, we introduce the decision problem of HWMCWP: Given a hybrid wireless mesh instance $(\bar{G} = (\bar{V}, \bar{E}), \tilde{G} = (\tilde{V}, \tilde{E}, \mathcal{C}), \mathcal{P})$, nodes s and t and a constant $K \in \mathbb{R}^+$, the task is to find a path from s to t that transports at least K units of flow through the hybrid network.

Theorem. *HWMCWP is NP-complete.*

Proof. First, HWMCWP's decision problem is $\in$ NP, because a nondeterministic algorithm guesses a subset of $\bar{E} \cup \tilde{E}$ and checks if these edges resemble a path with at least K units of flow. Checking is a bit more complex, because edges on the same channel require more attention as their achievable rates are equally shared if they are within interference range, but this can be clearly done in polynomial time.

To show NP-completeness, we reduce Path Avoiding Forbidden Pairs (PAFP) (which is NP-complete [10]) to HWMCWP. PAFP is defined as follows: Given a graph $G = (V, E)$, $s, t \in V$ and a set $F \subseteq \{\{a, b\} \mid a, b \in V, a \neq b\}$, PAFP asks if there is a path from s to t containing at most one node from each pair in F.

Let G, s, t, F be an arbitrary instance of PAFP. We need to construct an instance of HWMCWP $(\bar{G} = (\bar{V}, \bar{E}), \tilde{G} = (\tilde{V}, \tilde{E}, \mathcal{C}), \mathcal{P})$, $s, t \in \bar{V} \cup \tilde{V}$ and $K \in \mathbb{R}$ such that it has a widest path from s to t with at least K units of flow if and only if G has a path from s to t that contains at most one node from each pair in F.

A node of V not occurring in any pair of F will be in $\bar{V}$ without modifications. Also, an edge of E not incident to any node included in a pair of F will be in $\bar{E}$ without modifications. For each forbidden pair (a, b), we introduce a gadget $R_{a,b}$ consisting of four nodes $in^a_{a,b}$, $out^a_{a,b}$, $in^b_{a,b}$ and $out^b_{a,b}$ placed such that $(in^a_{a,b}, out^a_{a,b})$, $(in^b_{a,b}, out^b_{a,b}) \in \tilde{E}$ (i.e. within communication range), but all other tuples $\notin \tilde{E}$. This gadget is shown in Fig. 2. All wireless interfaces of nodes in $R_{a,b}$ are tuned to the same channel, therefore, edges in $R_{a,b}$ are correlated, when placed within correlation range. This can be easily achieved using a rectangular placement (by setting $\mathcal{P}$ appropriately). Please note that all gadgets must be placed in the same rectangular shape, i.e., achievable data rates and correlations will be exactly the same for every gadget. To prevent gadgets to correlate with each other, they use different channels (requiring sufficiently many channels) or they are placed far enough apart from each other, i.e., $\mathcal{C}$ and $\mathcal{P}$ need to be set accordingly.

Let $v \in V$ be a node appearing in at least one pair of F. Without loss of generality, we consider a fixed order of these pairs: $\{v, w_0\}, \ldots, \{v, w_{n-1}\}$. As described before, each pair $\{v, w_i\}$ will have its own gadget R_{v,w_i}, $0 \leq i < n$. We add edges $\{(out^v_{v,w_i}, in^v_{v,w_{i+1}}) \mid 0 \leq i < n-1\}$ to $\bar{E}$, which in conjunction with wireless edges from the gadgets form a path representing $v \in V$ in the HWMCWP instance. The path starts at in^v_{v,w_0} and ends at $out^v_{v,w_{n-1}}$ and is illustrated in Fig. 3. For any $(x, v) \in E$, an edge (x, in^v_{v,w_0}) will be in $\bar{E}$ and for any $(v, y) \in E$ an edge $(out^v_{v,w_{n-1}}, y)$ is added to $\bar{E}$.

Finally, setting K to the rate achievable between two wirelessly connected nodes in a gadget ($\tilde{r}$) finalizes the construction, which clearly requires polynomial time.

Now, suppose there exists a path p in G from s to t, which fulfills PAFP requirements. A solution for HWMCWP can be easily obtained by substituting edges and nodes of the path according to the construction seen above, resulting in a path p'. Because at most one node of each pair in F is used by p, at most one edge in each gadget is used. Therefore, the p' will be able to transport K units of flow and is the solution to HWMCWP.

Inversly, let p' be a path in the HWMCWP instance that transports K units of flow. As K units can be carried

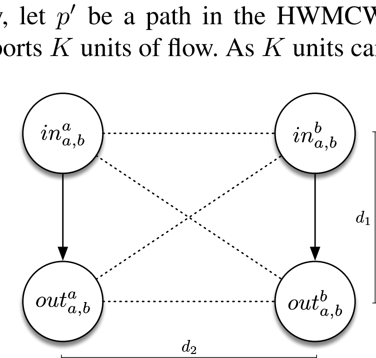

Fig. 2. Gadget $R_{a,b}$ representing a forbidden pair $(a, b) \in F$. Distance d_1 is less than communication range and d_2 is larger than communication range, but still less than interference range.

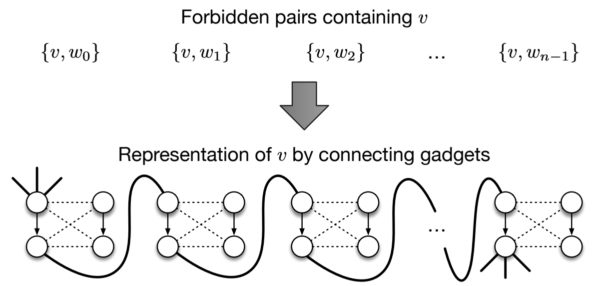

Fig. 3. The representation of a node v in the PAFP instance is a path connecting gadgets in the HWMCWP instance.

by p', we know that in each gadget at most one of two edges is used. If not, the resulting rate would be much lower than K, as edges within one gadget correlate with each other and need to share airtime. Therefore, p' can be transformed to a path p in G by simply applying the reverse construction for each node and edge in p. Because at most one edge of each gadget is used in p', it is easy to see that p fulfills PAFP requirements. $\square$

Please note that the proof requires an unrealistically large channel set and unreasonably long cables. Therefore, the practical impact of HWMCWP's NP-completeness may be insignificant, as the construction in the proof has very little resemblance with realistic topologies. However, optimal solutions to the problem may lead to increased computational complexity, which in turn justifies the need for heuristic solutions. Wired connections do not play a critical role in the proof's construction, which is why NP-completeness could be shown for non-hybrid topologies, too. However, they allow simplifications, i.e., wireless edges are only required within gadgets to cause interference. Nevertheless, wired edges could be omitted, e.g., with an even larger set of wireless channels.

B. Heuristic Solution

The proposed heuristic in this paper is to extend Dijkstra's algorithm. As explained in Sec. II, it already works with maximum rates instead of minimum costs thanks to some minor modifications whenever metrics are handled [1]. We briefly summarize these modifications: 1) The initial rate from s to itself is ∞, whereas initial rates to all other nodes are 0. 2) If a rate from s to an arbitrary node x during the algorithm is $r(x)$ and an edge $e = (x, y)$ with rate $r(e)$ is examined, then: $r(y) = \min\{r(x), r(e)\}$. 3) A node with the maximum known rate is chosen and processed prior to others.

Naturally, wired edges will not be affected from any further modifications, as they can be used independently from wireless connections. To consider multiple use of the same channel, we introduce a channel history H, which is a tuple containing recently used channels. It has fixed length and will be maintained using the FIFO principle. Depending on the currently inspected edge e' we need to consider the following cases: 1) If for the channel of a wireless edge e', i.e., $c(e')$, condition $c(e') \notin H$ holds, $c(e')$ is simply added to H. Further metric modifications are not required. 2) If $c(e')$ is already $k > 0$ times in the history, i.e., edges $e_0, \dots, e_{k-1}$ use the same channel, the metric for achievable data rate needs to be adapted. In the latter case, the metric value of e' is modified to be $\min\{r(e_i)/(k+1) \mid 0 \leq i < k\}$, reflecting multiple use of the same channel. The minimum of all affected edges is a plausible choice, since any of these edges could be a bottleneck. If (during the algorithm) the channel history exceeds its maximum length, oldest entries will be dropped (FIFO principle). Please note that this is only a heuristic, because we implicitly change rates of already processed edges, which violates Dijkstra's principles.

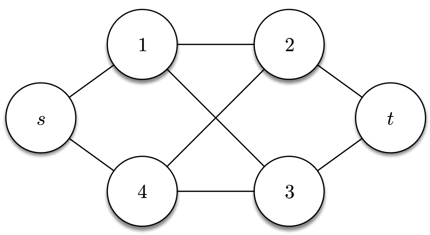

Fig. 4. Example instance of PAFP. Please note that displaying the corresponding HWMCWP instance would look too complex.

C. Shortcomings of the Heuristic

Due to NP-completeness of both PAFP and HWMCWP, there must be an instance of PAFP (and after transformation, one of HWMCWP) which cannot be optimally solved using the heuristic presented in Sec. IV-B. This example is shown in Fig. 4 and whether or not an optimal solution can be obtained depends on the processing order of the graph's nodes. Clearly: $(s, 4, 3, t)$ is a vaild path in the PAFP instance. Please note that after transformation, i.e., in the HWMCWP instance, $\forall\{a,b\} \in F$ gadgets $R_{a,b}$ need to be considered. Nodes from the PAFP instance will be translated into paths connecting gadgets, which are shown in Tab. I for better understanding. Our heuristic when started with the transformed HWMCWP starts in s, discovering $in^1_{1,2}$ and $in^4_{2,4}$ which can be reached with the wired rate $\bar{r}$. After passing through the gadgets, the rate is reduced to the wireless rate $\tilde{r}$. Therefore, $in^2_{1,2}$ and $in^3_{1,3}$ can be reached from both $out^1_{1,3}$ and $out^4_{2,4}$ with the same rate $\tilde{r}$. Suppose the algorithm examines the neighbors of $out^1_{1,3}$ next, then $in^2_{1,2}$ and $in^3_{1,3}$ will be reached with $\tilde{r}$ as well. Subsequently, the algorithm processes $in^2_{1,2}$ and $in^3_{1,2}$ (as well as nodes following the path), rendering them impossible to discover from $out^4_{2,4}$, as Dijkstra's algorithm does not allow already processed nodes to be revisited – which would be necessary to find the optimal path. Instead, the heuristic would stick to the choice it made earlier and would be forced to traverse the same gadget twice, hence, using edges interfering with each other.

TABLE I
NODE TRANSFORMATION OF PAFP INSTANCE (FIG. 4)

Node in PAFP Instance	**Path in HWMCWP Instance**
1	$in^1_{1,2}, out^1_{1,2}, in^1_{1,3}, out^1_{1,3}$
2	$in^2_{1,2}, out^2_{1,2}, in^2_{2,4}, out^2_{2,4}$
3	$in^3_{1,3}, out^3_{1,3}$
4	$in^4_{2,4}, out^4_{2,4}$

Therefore, it would yield a path transporting $\tilde{r}/2$ instead of $\tilde{r}$. The example could be easily modified to cause the heuristic's result to be arbitrarily small, therefore, the worst-case approximation error cannot be bounded.

V. CONCLUSION

In this paper, we examined the complexity of the widest path problem in hybrid multi-channel WMNs based on a simple, yet quite realistic model. We proved NP-completeness of the problem, despite the simplicity of the underlying model. Nevertheless, our heuristic design based on modifications to Dijkstra's algorithm has the potential for good results (when compared to heuristics for slotted time models), because realistic settings are quite different to pathological cases in which the heuristic inherently fails. We already demonstrated the heuristic's ability to provide near-optimal results on real-world-like topologies in certain use cases [5].

In future work, we will examine the widest path problems' complexity in wireless networks with a single channel, only. Furthermore, the heuristic's performance needs to be verified using packet-level simulations.

REFERENCES

[1] D. Medhi and K. Ramasamy, *Network Routing: Algorithms, Protocols, and Architectures*. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., 2007.

[2] Y. Zhu and B. Li, "Overlay Networks with Linear Capacity Constraints," *IEEE Transactions on Parallel and Distributed Systems*, vol. 19, no. 2, pp. 159–173, 2008.

[3] K. Jain, J. Padhye, V. N. Padmanabhan, and L. Qiu, "Impact of Interference on Multi-Hop Wireless Network Performance," *Wireless Networks*, vol. 11, pp. 471–487, 2005.

[4] M. Saad, "Joint Optimal Routing and Power Allocation for Spectral Efficiency in Multihop Wireless Networks," *IEEE Tr. on Wireless Communications*, vol. 13, no. 5, pp. 2530–2539, 2014.

[5] M. Backhaus, M. Rossberg, and G. Schaefer, "Modeling Maximum Flow and Widest Path in Hybrid Multi-Channel Wireless Mesh Networks," in *Intl. Conf. on Performance Evaluation and Modeling in Wired and Wireless Networks*, 2020, currently under review. [Online]. Available: tu-ilmenau.de/telematik/mfwpwmn

[6] J. Thomas, V. Noufidali, and F. Jose, "A Survey on Bandwidth Guaranteed Routing Techniques in Wireless Mesh Networks," *Intl. Journal of Computer Applications*, vol. 60, pp. 11–15, 2012.

[7] R. Hou, K. Lui, F. Baker, and J. Li, "Hop-by-Hop Routing in Wireless Mesh Networks with Bandwidth Guarantees," *IEEE Trans. on Mobile Computing*, vol. 11, no. 2, pp. 264–277, 2012.

[8] M. Curado and E. Monteiro, "A Survey of QoS Routing Algorithms," in *Intl. Conference on Information Technology*, 2004.

[9] T. Cormen, C. Leiserson, and R. Rivest, *Introduction to Algorithms*. MIT Press, 1990.

[10] H. N. Gabow, S. N. Maheshwari, and L. J. Osterweil, "On Two Problems in the Generation of Program Test Paths," *IEEE Trans. on Software Engineering*, vol. SE-2, no. 3, pp. 227–231, 1976.