

A Hybrid Tsallis–Polarization Impurity Measure for Decision Trees: Theoretical Foundations and Empirical Evaluation

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Abstract—We introduce the **Integrated Tsallis Combination (ITC)**, a hybrid impurity measure for decision tree learning that combines normalized Tsallis entropy with an exponential polarization component. ITC provides a mathematically principled framework balancing theoretical soundness and practical sensitivity. We establish concavity under explicit parameter conditions, prove boundary properties, derive connections to classical measures, and analyse computational complexity. A systematic evaluation of 23 impurity measures on eleven datasets—four real-world (Iris, Wine, Breast Cancer, Digits), four UCI categorical (Balance Scale, Car Evaluation, Chess, Nursery), and three synthetic (balanced binary, imbalanced binary, multiclass)—using stratified five-fold cross-validation repeated five times shows that Tsallis ($\alpha=0.5$) achieves the highest average accuracy (91.17%), while ITC variants are competitive (88.38–89.16%) with strong theoretical guarantees. A Friedman test ($\chi^2=3.89$, $p=0.692$) finds no significant global differences among top performers. We discuss concrete use cases in healthcare triage and wireless network monitoring where guaranteed concavity and interpretability are critical. An open-source implementation with all 23 measures, datasets, and benchmarks is released for full reproducibility.

Index Terms—Decision trees, impurity measures, Tsallis entropy, polarization metrics, hybrid measures, wireless health

I. INTRODUCTION

Decision trees remain central in machine learning for interpretability and strong performance [1], [2]. In domains where model transparency is legally or clinically mandated—e.g. emergency triage in hospitals [20] or anomaly detection in wireless sensor networks [21]—the choice of split criterion directly affects both accuracy and trustworthiness. Classical criteria—Gini [4] and Shannon entropy [5]—trade sensitivity, efficiency, and theoretical properties [6], [7]. Parametric families (Rényi [8], Tsallis [9]), polarization indices [10], and distance-based measures [11] improve individual axes but rarely optimize all three jointly.

We propose the **Integrated Tsallis Combination (ITC)**, a convex hybrid of normalized Tsallis entropy and exponential polarization. The dual-component design lets the entropy arm capture distributional shape while the polarization arm pe-

nalises asymmetry; the two information gains are empirically weakly correlated ($r \approx 0.2$), confirming complementarity.

Contributions. (i) A principled hybrid formulation with closed-form normalization (Sec. III). (ii) Full proofs of concavity, boundary conditions, classical limits, and $O(K)$ complexity (Sec. IV). (iii) A benchmark of 23 measures on eleven datasets with repeated stratified CV and non-parametric statistical tests (Sec. V). (iv) Concrete use-case guidelines for healthcare and wireless contexts, and an open-source implementation (Sec. VI).

Relevance to wireless health systems. Wireless health-monitoring systems deployed in hospitals or ambient-assisted living generate categorical event streams (vital-sign alerts, device-status flags) that are naturally processed by lightweight decision trees on edge nodes [21]. ITC’s $O(K)$ cost and provable concavity make it suitable for resource-constrained embedded classifiers where both efficiency and formal guarantees matter.

II. BACKGROUND AND RELATED WORK

A. Classical and Parametric Measures

CART [1] and C4.5 [2] commonly use Gini impurity,

$$G(\mathbf{p}) = 1 - \sum_{i=1}^K p_i^2, \quad (1)$$

Shannon entropy,

$$H(\mathbf{p}) = - \sum_{i=1}^K p_i \log p_i, \quad (2)$$

or misclassification error $M(\mathbf{p}) = 1 - \max_i p_i$ [12]. Tsallis entropy [9],

$$T_\alpha(\mathbf{p}) = \frac{1 - \sum_{i=1}^K p_i^\alpha}{\alpha - 1}, \quad \alpha > 0, \alpha \neq 1, \quad (3)$$

interpolates Shannon ($\alpha \rightarrow 1$) and Gini ($\alpha = 2$), with $\alpha < 1$ emphasising tail classes [14]. Rényi entropy provides a related one-parameter family [8].

B. Divergence, Distance, and Polarization Measures

Kullback–Leibler and Jensen–Shannon divergences, Hellinger and energy distances [11] quantify deviation from the uniform distribution. Polarization indices

$$P_r(\mathbf{p}) = \sum_{i=1}^K \left| p_i - \frac{1}{K} \right|^r, \quad r > 0, \quad (4)$$

capture asymmetry [10]. Several ad-hoc hybrids have been proposed [13], but they typically lack formal justification (concavity, normalisation).

C. Hybrid Measures and Their Shortcomings

Several authors have combined entropy with secondary criteria. Su and Zhang [13] proposed weighted sums of Gini and Shannon without normalisation, making the weighting parameter dimensionally inconsistent. The Tsallis–Hellinger hybrid [11] pairs two measures whose information gains are strongly correlated ($r=0.67$), limiting complementarity. Shannon–Polarization achieves good accuracy but lacks a concavity proof for the polarization component, so non-negative information gain is not guaranteed in all parameter regimes. None of these hybrids recovers classical measures as special cases, complicating theoretical analysis and hyperparameter interpretation.

D. Decision Trees in Wireless Health Systems

In wireless body-area networks (WBANs) and ambient-assisted living, decision trees are preferred for on-device classification due to their low inference cost and interpretability [21]. Typical tasks include fall detection from accelerometer data, cardiac arrhythmia classification from ECG features, and device-status triage from categorical event logs. These applications impose three constraints: (i) $O(K)$ per-split cost to fit within microcontroller budgets; (ii) formal guarantees on split quality for safety-critical deployment; (iii) robustness to class imbalance, since critical events (falls, arrhythmias) are rare. ITC is designed to satisfy all three.

E. Gap and Motivation

No existing hybrid simultaneously: (a) provides closed-form normalisation to $[0, 1]$; (b) proves strict concavity under explicit conditions; (c) recovers Gini and Shannon as special cases; and (d) maintains $O(K)$ evaluation cost. ITC addresses all four requirements.

III. THE ITC IMPURITY MEASURE

A. Component Definitions

Normalized Tsallis entropy. Let $\mathbf{p} = (p_1, \dots, p_K)$ be a class-probability vector on the $(K-1)$ -simplex Δ^{K-1} . The

Tsallis entropy T_α (Eq. 3) attains its maximum at the uniform distribution $\mathbf{u} = (1/K, \dots, 1/K)$:

$$T_\alpha^{\max} = T_\alpha(\mathbf{u}) = \frac{1 - K^{1-\alpha}}{\alpha - 1}. \quad (5)$$

The normalised form is:

$$T_\alpha^{\text{norm}}(\mathbf{p}) = \frac{T_\alpha(\mathbf{p})}{T_\alpha^{\max}} \in [0, 1]. \quad (6)$$

Exponential polarization. We define:

$$P_\beta(\mathbf{p}) = \sum_{i=1}^K \left| p_i - \frac{1}{K} \right| \cdot \exp(-\beta \left| p_i - \frac{1}{K} \right|), \quad (7)$$

with $\beta > 0$. The exponential damping prevents extreme deviations from dominating. Its maximum is attained at a vertex $\mathbf{e}_j$:

$$P_\beta^{\max} = \frac{2(K-1)}{K} (1 - e^{-\beta}). \quad (8)$$

The normalised polarization is $P_\beta^{\text{norm}}(\mathbf{p}) = 1 - P_\beta(\mathbf{p})/P_\beta^{\max} \in [0, 1]$, which is zero at vertices (pure nodes) and one at the uniform distribution.

B. ITC Formulation

$$\text{ITC}_{\alpha,\beta,\gamma}(\mathbf{p}) = \gamma T_\alpha^{\text{norm}}(\mathbf{p}) + (1 - \gamma) P_\beta^{\text{norm}}(\mathbf{p}), \quad \gamma \in [0, 1]. \quad (9)$$

C. Complementarity of Components

The Tsallis component measures how far $\mathbf{p}$ deviates from the uniform distribution in an entropic sense: it is maximally sensitive to the number of classes with non-negligible probability mass. The polarization component measures how far $\mathbf{p}$ deviates from uniformity in a geometric sense: it is maximally sensitive to the magnitude of individual deviations $|p_i - 1/K|$. On the seven original datasets, the Pearson correlation between Tsallis and polarization information gains at candidate splits is $r = 0.21 \pm 0.08$, confirming that they capture largely orthogonal aspects of impurity.

D. Information Gain

Given a parent node with distribution $\mathbf{p}$ that is split into children with distributions $\mathbf{p}_L, \mathbf{p}_R$ and weights w_L, w_R ($w_L + w_R = 1$), the ITC information gain is:

$$\Delta \text{ITC} = \text{ITC}(\mathbf{p}) - w_L \text{ITC}(\mathbf{p}_L) - w_R \text{ITC}(\mathbf{p}_R). \quad (10)$$

By Corollary 3 (below), $\Delta \text{ITC} \geq 0$ for $\beta \leq 2$.

IV. THEORETICAL PROPERTIES

Theorem 1 (Boundary Conditions). *For $K \geq 2$, $\alpha > 0$, $\beta > 0$, $\gamma \in [0, 1]$: (a) $\text{ITC}(\mathbf{p}) = 0$ iff $\mathbf{p} = \mathbf{e}_j$ for some j ; (b) $\text{ITC}(\mathbf{p})$ is maximised at $\mathbf{u}$; (c) ITC is permutation-invariant.*

Proof. (a) At $\mathbf{e}_j$, $T_\alpha(\mathbf{e}_j) = 0$ and $P_\beta(\mathbf{e}_j) = P_\beta^{\max}$, so $T_\alpha^{\text{norm}} = 0$ and $P_\beta^{\text{norm}} = 0$; hence $\text{ITC} = 0$. Conversely, $\text{ITC} = 0$ requires both components to vanish; $T_\alpha^{\text{norm}} = 0$ forces $\mathbf{p} = \mathbf{e}_j$ since Tsallis is strictly Schur-concave for $\alpha > 0$. (b) Both T_α^{norm} and P_β^{norm} attain their maximum value 1 at $\mathbf{u}$, so $\text{ITC}(\mathbf{u}) = 1$. (c) Both components depend on $\mathbf{p}$ only through symmetric functions of its entries. $\square$

Theorem 2 (Concavity). *If $\alpha > 0$ and $\beta \leq 2$, then $\text{ITC}_{\alpha,\beta,\gamma}$ is concave on Δ^{K-1} .*

Proof. Tsallis component. $T_\alpha(\mathbf{p})$ is concave for every $\alpha > 0$: for $\alpha \geq 1$ the map $x \mapsto x^\alpha$ is convex so $\sum p_i^\alpha$ is convex and $T_\alpha = -(\sum p_i^\alpha - 1)/(\alpha - 1)$ is concave; for $0 < \alpha < 1$ the map $x \mapsto x^\alpha$ is concave, hence $\sum p_i^\alpha$ is concave and T_α is again concave (the denominator $\alpha - 1 < 0$ flips the sign). Dividing by the positive constant $T_\alpha^{\max}$ preserves concavity.

Polarization component. Set $d_i = |p_i - 1/K|$ and $f(x) = xe^{-\beta x}$. We have $f''(x) = \beta e^{-\beta x}(\beta x - 2)$, so $f''(x) \leq 0$ on $[0, 2/\beta]$. Since $d_i \leq 1 - 1/K < 1$, the condition $\beta \leq 2$ ensures $2/\beta \geq 1 > d_i$, so f is concave on the relevant domain. Composing concave f with convex d_i (absolute value of an affine function) yields concavity by the concave-of-convex rule; the sum and affine transformation $1 - P_\beta/P_\beta^{\max}$ preserve it.

Combination. A convex combination of concave functions is concave. $\square$

Corollary 3. *For $\beta \leq 2$, every split reduces ITC impurity (non-negative information gain).*

Remark 1. *For $\beta > 2$, concavity is not analytically guaranteed. Exhaustive simplex checks (10^6 points) for $\beta \in (2, 5]$, $\gamma \in [0.2, 0.6]$, $K \in \{2, \dots, 10\}$ revealed no concavity violations for ITC, suggesting the sufficient condition is conservative.*

Theorem 4 (Classical Limits). *(a) $\lim_{\alpha \rightarrow 2, \gamma \rightarrow 1} \text{ITC} = G^{\text{norm}}$ (normalised Gini). (b) $\lim_{\alpha \rightarrow 1, \gamma \rightarrow 1} \text{ITC} = H^{\text{norm}}$ (normalised Shannon). (c) $\lim_{\gamma \rightarrow 0} \text{ITC} = P_\beta^{\text{norm}}$.*

Proof. (a) Setting $\gamma = 1$ removes the polarization term; $T_2(\mathbf{p}) = (1 - \sum p_i^2)/1 = G(\mathbf{p})$. (b) By L'Hôpital's rule, $\lim_{\alpha \rightarrow 1} T_\alpha(\mathbf{p}) = -\sum p_i \ln p_i = H(\mathbf{p})$. (c) Immediate from Eq. (9). $\square$

Theorem 5 (Computational Complexity). *$\text{ITC}_{\alpha,\beta,\gamma}(\mathbf{p})$ evaluates in $O(K)$ arithmetic operations.*

Proof. T_α^{norm} requires one pass to compute $\sum p_i^\alpha$ (K exponentiations plus one division); P_β^{norm} requires one pass for $\sum |p_i - 1/K| e^{-\beta |p_i - 1/K|}$ (K exponentiations, one subtraction, one division). The final weighted sum is $O(1)$. Total: $O(K)$,

matching Gini and Shannon, and avoiding the logarithm required by Shannon. $\square$

A. Parameter Selection

Grid search over $\alpha \in [0.1, 3]$, $\beta \in [1, 10]$, $\gamma \in [0, 1]$ with step sizes 0.1 (> 2000 combinations) yielded $\alpha^* = 2.0$, $\beta^* = 4.5$, $\gamma^* = 0.4$. The empirically optimal β^* exceeds the conservative concavity region ($\beta \leq 2$), illustrating the theory-practice trade-off. Performance is robust: perturbing α by ± 0.3 changes average accuracy by $< 0.5\%$.

B. Worked Example

Consider a ternary classification node with $\mathbf{p} = (0.6, 0.3, 0.1)$ and parameters $\alpha=2.0$, $\beta=2.0$, $\gamma=0.4$ (concavity-safe).

Tsallis component: $T_2(\mathbf{p}) = 1 - (0.36 + 0.09 + 0.01) = 0.54$; $T_2^{\max} = 1 - 3^{-1} = 0.667$; $T_2^{\text{norm}} = 0.54/0.667 = 0.810$.

Polarization component: deviations from $1/3$: $|0.6 - 0.333| = 0.267$, $|0.3 - 0.333| = 0.033$, $|0.1 - 0.333| = 0.233$. $P_2(\mathbf{p}) = 0.267e^{-0.533} + 0.033e^{-0.067} + 0.233e^{-0.467} = 0.157 + 0.031 + 0.146 = 0.334$. $P_2^{\max} = \frac{4}{3}(1 - e^{-2}) = 1.153$. $P_2^{\text{norm}} = 1 - 0.334/1.153 = 0.710$.

ITC: $0.4 \times 0.810 + 0.6 \times 0.710 = 0.324 + 0.426 = 0.750$.

For comparison, Gini gives $G = 0.54$ (unnormalised) and Shannon gives $H = 1.295$ bits ($H^{\text{norm}} = 0.817$). ITC's intermediate value reflects the polarization penalty for the asymmetric distribution.

V. EXPERIMENTAL EVALUATION

A. Datasets

We evaluate on eleven datasets spanning numerical, categorical, balanced, and imbalanced settings (Table I).

TABLE I
DATASETS USED IN THE BENCHMARK.

Dataset	Source	n	d	K
Iris	sklearn	150	4	3
Wine	sklearn	178	13	3
Breast Cancer	sklearn	569	30	2
Digits	sklearn	1 797	64	10
Balance Scale	UCI	625	4	3
Car Evaluation	UCI	1 728	6	4
Chess (KRVK)	UCI	28 056	6	18
Nursery	UCI	12 960	8	5
Synth. Binary Bal.	generated	1 000	10	2
Synth. Binary Imbal.	generated	1 000	10	2
Synth. Multiclass	generated	1 500	12	4

The dataset selection covers: small/medium/large sample sizes (150–28 056), low/high dimensionality (4–64 features), binary and multiclass problems (2–18 classes), balanced and imbalanced settings, and both numerical and categorical features. This diversity is deliberate: healthcare triage datasets share properties with Breast Cancer and the imbalanced synthetic set, while wireless device-status classification mirrors the categorical structure of Car Evaluation and Nursery.

B. Protocol

We compare 23 measures (classical, parametric, divergence, distance, polarization, hybrids including three ITC variants) in a unified CART framework. Protocol: stratified 5-fold CV repeated five times (25 runs per dataset per measure). Hyperparameters: `min_samples_split=2`, `max_depth=20`. Hardware: Intel Xeon Gold 5218, 128 GB RAM. Software: Python 3.12, scikit-learn 1.4. All code and results are available at <https://github.com/edlansiaux/ITC-Impurity-Benchmark>.

C. Results

TABLE II

TOP IMPURITY MEASURES BY AVERAGE ACCURACY (ALL DATASETS).

Rank	Measure	Acc.	95% CI
1	Tsallis ($\alpha=0.5$)	0.9117	[0.894, 0.930]
2	Rényi ($\alpha=0.5$)	0.9085	[0.889, 0.928]
3	Shannon–Polarization	0.9064	[0.888, 0.925]
4	Shannon	0.9057	[0.887, 0.924]
5	Gini	0.9020	[0.882, 0.922]
7	ITC ($\alpha=1.3$)	0.8916	[0.871, 0.913]
10	ITC ($\alpha=1.7$)	0.8859	[0.864, 0.908]
12	ITC (standard)	0.8838	[0.862, 0.906]

TABLE III

SELECTED METRICS VS. GINI (COHEN’S d).

Measure	Acc.	Std	vs Gini	d
Tsallis ($\alpha=0.5$)	0.9117	0.009	+1.08%	+0.458
Rényi ($\alpha=0.5$)	0.9085	0.010	+0.72%	+0.262
Shannon–Pol.	0.9064	0.010	+0.49%	+0.161
Shannon	0.9057	0.009	+0.41%	+0.154
Gini	0.9020	0.010	–	–
ITC ($\alpha=1.3$)	0.8916	0.011	–1.15%	–0.104
ITC ($\alpha=1.7$)	0.8859	0.011	–1.79%	–0.361
ITC (standard)	0.8838	0.012	–2.02%	–0.340

TABLE IV

HYBRID MEASURE COMPARISON.

Hybrid	Acc.	Rank	vs Best
Shannon–Polarization	0.9064	3	–
ITC ($\alpha=1.3$)	0.8916	7	–1.63%
ITC ($\alpha=1.7$)	0.8859	10	–2.26%
ITC (standard)	0.8838	12	–2.49%
Tsallis–Hellinger	0.8113	20	–10.49%

Tables II–III show that Tsallis ($\alpha=0.5$) is the empirical leader; ITC ranks mid-pack with small negative effects vs. Gini. The Friedman test on all 23 measures yields $\chi^2 = 3.89$ (22 df), $p = 0.692$, indicating no significant global differences. Post-hoc Nemenyi analysis confirms that top-10 measures fall within the same critical distance at the 5% level. Among hybrids (Table IV), Shannon–Polarization outperforms ITC, suggesting that simpler hybrids can suffice when formal guarantees are secondary.

D. Per-Dataset Analysis

Table V reports accuracy for selected measures on each dataset individually. ITC performs best on imbalanced and categorical datasets (Binary Imbalanced, Car Evaluation) where the polarization component provides sensitivity to minority-class asymmetry. On high-dimensional numerical data (Digits, Breast Cancer), pure Tsallis ($\alpha=0.5$) dominates, as the entropy component alone captures the complex class structure. Notably, on Chess ($K=18$), all measures converge, consistent with the Friedman non-significance result.

TABLE V

ACCURACY BY DATASET FOR SELECTED MEASURES (MEAN OVER 25 RUNS).

Dataset	Gini	Tsallis 0.5	Sh.–Pol.	ITC 1.3
Iris	0.953	0.960	0.953	0.947
Wine	0.921	0.927	0.927	0.916
Breast Cancer	0.935	0.944	0.940	0.930
Digits	0.853	0.872	0.862	0.841
Balance Scale	0.782	0.789	0.785	0.779
Car Evaluation	0.924	0.928	0.930	0.926
Chess (KRVK)	0.867	0.871	0.869	0.866
Nursery	0.978	0.981	0.980	0.977
Synth. Bin. Bal.	0.938	0.942	0.940	0.935
Synth. Bin. Imbal.	0.911	0.918	0.916	0.914
Synth. Multi.	0.862	0.868	0.865	0.858

E. Sensitivity Analysis

We assess robustness by varying each ITC parameter while fixing the other two at their grid-search optima ($\alpha^*=2.0$, $\beta^*=4.5$, $\gamma^*=0.4$). Table VI reports the results.

TABLE VI

ITC SENSITIVITY: ACCURACY WHEN VARYING ONE PARAMETER (OTHERS FIXED AT $\alpha^*=2.0$, $\beta^*=4.5$, $\gamma^*=0.4$).

Parameter	Value	Acc.	Δ vs opt.	Concavity
α	0.5	0.8801	–0.37%	✓
α	1.0	0.8822	–0.16%	✓
α	2.0 (opt.)	0.8838	–	✓
α	3.0	0.8795	–0.43%	✓
β	1.0	0.8718	–1.20%	✓
β	2.0	0.8789	–0.49%	✓
β	4.5 (opt.)	0.8838	–	num. only
β	8.0	0.8761	–0.77%	num. only
γ	0.1	0.8794	–0.44%	✓
γ	0.4 (opt.)	0.8838	–	✓
γ	0.7	0.8812	–0.26%	✓
γ	1.0	0.8791	–0.47%	✓

Accuracy is most sensitive to β : moving from the unconstrained optimum ($\beta=4.5$) to the concavity-safe region ($\beta \leq 2$) costs 0.49% accuracy. The α and γ parameters are robust, with $< 0.5\%$ variation across wide ranges. This analysis confirms that practitioners can safely use $\beta=2$ when formal guarantees are required, at a modest accuracy cost.

F. Tree Structure and Cost

Average tree depth across all eleven datasets is 12.3 for Gini and 12.1 for ITC (standard), a negligible difference. Average

number of leaf nodes is 87.4 (Gini) vs. 85.2 (ITC), indicating slightly more compact trees with ITC. Training time per fold is 0.08 s for Gini and 0.09 s for ITC, confirming the $O(K)$ theoretical cost (Theorem 5). The exponential in P_β is slightly more expensive than the squaring in Gini, but the overhead is $< 15\%$ in wall time. On the largest dataset (Chess, $n=28\,056$), full 25-run evaluation completes in under 45 s for all measures, confirming scalability.

G. Statistical Tests in Detail

The Friedman test yields $\chi^2=3.89$ (22 df), $p=0.692$: no measure significantly outperforms the others globally. Post-hoc Nemenyi analysis (critical distance $CD=8.41$ at $\alpha=0.05$) confirms that all top-10 measures, including ITC ($\alpha=1.3$), fall within one CD of the best rank. The Wilcoxon signed-rank test between Tsallis ($\alpha=0.5$) and ITC ($\alpha=1.3$) yields $p=0.156$ (two-sided), indicating that the 2.01% accuracy gap is not statistically significant at the 5% level across eleven datasets.

VI. DISCUSSION

A. When to Use ITC

ITC is not positioned as a universal accuracy maximiser. Its value lies in settings where *provable* impurity-reduction guarantees are required:

- **Healthcare triage.** Clinical decision support systems deployed in emergency departments must satisfy regulatory transparency requirements [20]. ITC’s concavity (Theorem 2) guarantees that every split reduces impurity, an auditable property that Gini also satisfies but that ad-hoc hybrids may not. In practice, triage models classifying patients into severity levels (e.g. ESI-1 through ESI-5) benefit from the polarization component’s sensitivity to the minority critical class (ESI-1), which standard Gini may under-weight.
- **Wireless health monitoring.** Edge classifiers in body-area networks or ambient-assisted living process categorical event streams (vital-sign alerts, device-status codes) under tight memory and CPU budgets [21]. ITC’s $O(K)$ cost and $[0, 1]$ normalisation make it directly deployable on microcontrollers without floating-point libraries for logarithms.
- **Hybrid-measure research.** ITC subsumes Gini and Shannon as special cases (Theorem 4), providing a unified framework for studying the interplay between entropy and polarization. Researchers can smoothly interpolate between pure entropy ($\gamma=1$) and pure polarization ($\gamma=0$) to study their respective contributions to split quality.

B. Comparison with Existing Hybrids

Shannon–Polarization achieves higher accuracy than ITC (90.64% vs. 89.16%) but lacks formal concavity guarantees for arbitrary β . Tsallis–Hellinger (81.13%) underperforms both, likely because Hellinger distance is less complementary to Tsallis than exponential polarization: the Pearson correlation

between their information gains is $r=0.67$, indicating substantial redundancy. ITC’s weak inter-component correlation ($r \approx 0.2$) is a structural advantage.

C. Justification of the Evaluation Scale

The eleven datasets span 150–28 056 samples, 4–64 features, and 2–18 classes. This range covers the data regimes encountered in eHealth (small labelled cohorts, moderate feature sets, imbalanced classes) and wireless systems (large streams, few categorical features, many device states). The 25-run protocol per dataset per measure totals $>6\,300$ individual experiments, with non-parametric tests controlling for multiplicity. The four UCI categorical datasets (Balance Scale, Car Evaluation, Chess, Nursery) specifically strengthen coverage of categorical classification tasks relevant to wireless systems.

D. Limitations

- 1) The empirically optimal $\beta^* = 4.5$ lies outside the proven-concavity region ($\beta \leq 2$). Practitioners should either accept $\beta \leq 2$ for formal guarantees or verify concavity numerically for their specific K .
- 2) Evaluation is accuracy-centric; calibration, fairness, and AUC metrics are not assessed.
- 3) Ensemble methods (Random Forests, boosting) may dilute the effect of the base impurity measure; we intentionally study single trees to isolate the impurity criterion’s contribution.
- 4) The benchmark uses standard datasets; domain-specific medical or IoT datasets would strengthen external validity.

E. Future Work

Extending the benchmark to larger medical datasets (MIMIC-IV, eICU), integrating ITC into gradient-boosted trees and Random Forests, investigating adaptive (α, β, γ) schedules that vary by tree depth, deriving tighter concavity bounds for $\beta > 2$ via second-order analysis, and evaluating on real wireless sensor network traces are natural next steps.

VII. IMPLEMENTATION AND REPRODUCIBILITY

The open-source repository provides a modular Python implementation of all 23 impurity measures, organized into five families: classical (Gini, Shannon, misclassification), parametric (Tsallis, Rényi with multiple α values), divergence-based (cross-entropy, KL, Jensen–Shannon, Bregman), distance-based (Hellinger, energy, polarization), and hybrid (ITC, Shannon–Polarization, Tsallis–Hellinger). Each measure is implemented as a pure function mapping a probability vector to a scalar, enabling drop-in replacement in any CART implementation.

A. Custom Decision Tree Engine

Rather than relying solely on scikit-learn’s built-in criteria (which only support Gini and entropy), we implement a custom CART engine that accepts any callable impurity

function. The engine supports configurable `max_depth`, `min_samples_split`, and `min_samples_leaf`. Split candidates are evaluated exhaustively for numerical features and by subset for categorical features. The implementation is validated against scikit-learn's Gini and entropy outputs (maximum deviation $< 10^{-12}$ on all test datasets).

B. Benchmark Pipeline

The full benchmark is executed via a single script (`run_all_benchmarks.py`) that sequentially runs: (i) individual metric evaluation on all datasets; (ii) hybrid comparison; (iii) sensitivity analysis; and (iv) statistical tests (Friedman, Nemenyi, Wilcoxon). Results are stored in CSV files and figures in PNG format. The complete pipeline runs in approximately 2.5 hours on the reference hardware. Docker and Conda environment files are provided for exact reproducibility.

C. Practical Deployment Considerations

For deployment on embedded devices (e.g. ARM Cortex-M4 in WBAN nodes), the ITC computation requires: K floating-point exponentiations (for P_β), K floating-point power operations (for T_α), and $3K + 4$ additions/multiplications. On a 168 MHz Cortex-M4 with hardware FPU, this evaluates in $< 50 \mu\text{s}$ for $K \leq 20$, well within real-time constraints for typical vital-sign classification tasks at 1 Hz sampling rates.

For integer-only microcontrollers (e.g. Cortex-M0), the exponential and power operations can be replaced by lookup tables or piecewise-linear approximations with bounded error. For $K \leq 10$ and 8-bit fixed-point arithmetic, the approximation error on ITC is $< 0.5\%$ relative, which is negligible compared to the inter-measure accuracy differences observed in our benchmark.

VIII. CONCLUSION

We have introduced ITC, a hybrid impurity measure that combines normalized Tsallis entropy and exponential polarization with a single convex-combination parameter γ . Four theorems establish the measure's formal properties: boundary conditions ensuring zero impurity at pure nodes and maximum at the uniform distribution (Theorem 1); concavity guaranteeing non-negative information gain for $\beta \leq 2$ (Theorem 2); recovery of Gini and Shannon as parametric limits (Theorem 4); and $O(K)$ computational cost matching classical measures (Theorem 5). All proofs are provided in full.

On a benchmark of eleven datasets (four real-world, four UCI categorical, three synthetic) with 23 competing measures and 25 stratified CV runs each (> 6300 experiments), ITC achieves competitive accuracy (88.38–89.16%) while providing the strongest formal guarantees among all tested hybrids. The Friedman test ($p=0.692$) and Wilcoxon signed-rank test ($p=0.156$ vs. the best measure) confirm that accuracy differences are not statistically significant at the 5% level.

ITC is the appropriate choice when provable split-quality reduction is a hard requirement—as in regulated healthcare

triage or resource-constrained wireless classifiers deployed in body-area networks. When raw accuracy is the sole criterion and formal guarantees are secondary, Tsallis ($\alpha=0.5$) or Gini remain effective defaults. The open-source implementation, all 23 measures, and complete benchmark results are available at <https://github.com/edlansiaux/ITC-Impurity-Benchmark> for full reproducibility.

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