

Design SNR Optimization of Polar Codes for Successive Cancellation List Decoding

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Abstract—Polar codes are commonly constructed by identifying reliable bit channels under successive cancellation (SC) decoding for a fixed channel signal-to-noise ratio (SNR), which is used as a *design SNR*. However, such an SC-oriented construction is not necessarily optimal for successive cancellation list (SCL) decoding with a finite list size. This paper studies the design SNR as a parameter for optimizing finite-list SCL decoding and proposes a low-complexity design SNR selection method. Candidate information sets are constructed over the binary-input additive white Gaussian noise (BI-AWGN) channel by a conventional SC-oriented method, while their suitability for the given list size is evaluated over the binary erasure channel (BEC) by combining a minimum-weight union bound (MWUB) with an approximate list saturation probability newly introduced in this paper. Numerical results show that the proposed method recovers most of the block error rate (BLER) improvement achievable by exhaustive design SNR search for moderate and large list sizes without Monte Carlo simulations.

I. INTRODUCTION

Polar codes [1] are known to approach the channel capacity of symmetric binary-input memoryless channels with simple successive cancellation (SC) decoding as the code length increases. Channel polarization induced by polar codes transforms independent copies of a communication channel into polarized channels, i.e., a set of noisy channels and a set of noiseless channels. The most reliable (noiseless) bit channels will be used for information transmission, whereas the remaining (noisy) channels are not used (frozen). Therefore, polar code construction is equivalent to the identification of the reliabilities of the synthetic bit channels.

For a practical binary-input additive white Gaussian noise (BI-AWGN) channel, the reliability of the bit channels can be estimated by density evolution [2], but its computational complexity for accurate estimation increases substantially as the code length increases. Therefore, low-complexity algorithms such as Gaussian approximation (GA) [3] and reciprocal channel approximation (RCA) [4] have been developed. These constructions attempt to identify reliable bit channels from a given channel signal-to-noise ratio (SNR) parameter, often called a *design SNR*.

For practical finite code lengths, however, SC decoding often results in insufficient block error rate (BLER) performance. To address this problem, more powerful decoders have been proposed, such as successive cancellation list (SCL) decoding [5]. Since the conventional constructions developed for SC decoding are not necessarily well matched to SCL decoding, several alternative construction methods have been

developed. Construction methods in conjunction with dynamic frozen bits have been studied in [6]–[8], whereas those for polarization-adjusted convolutional (PAC) codes [9] have been investigated in [10], [11]. However, these construction methods often rely on heuristic parameter optimization to achieve favorable BLER performance. Machine learning-based constructions have also been developed in [12]–[15]. They achieve better BLER performance than the conventional constructions, but most of them rely on computationally demanding Monte Carlo simulations to obtain evaluation metrics such as BLER. Recently, the list size required for SCL decoding to achieve maximum-likelihood (ML) decoding has been discussed in several studies [16]–[18], and information-theoretic bounds on the required list size have been used for polar code construction. However, the available bounds can be loose for practical code parameters and do not directly characterize the BLER of SCL decoding with a prescribed finite list size. Therefore, a low-complexity criterion is still required to select a polar code for a specific operating SNR and list size.

This paper develops a design SNR optimization framework for finite-list SCL decoding motivated by [19], where the *interpolation* of polar codes to Reed-Muller (RM) codes is considered by increasing the design SNR. A straightforward approach to identifying the best design SNR is to construct a polar code and perform Monte Carlo simulations for each candidate design SNR, and then select the one that minimizes the BLER. However, this exhaustive BLER-based search is computationally expensive since it requires actual simulations with SCL decoding. Consequently, low-complexity selection method should be developed.

To this end, we propose a binary erasure channel (BEC)-aided design SNR selection method. We consider two error mechanisms: a final path selection error in the case that the correct path remains in the list, and removal of the correct path from the list during SCL decoding. We then consider the tradeoff between these two error mechanisms, both of which should be suppressed. The former is represented by a minimum-weight union bound (MWUB) term, which serves as a metric for the distance-limited ML decoding behavior. On the other hand, the latter is difficult to analyze, especially for the BI-AWGN channel, since error propagation directly affects the decoder behavior. Therefore, we focus on the BEC, where the synthetic channel erasure probabilities can be computed exactly. By taking advantage of the analytical simplicity of the BEC, we introduce a low-complexity measure

that accounts for the correct path removal due to list saturation. By combining the MWUB and this measure, we construct a metric for evaluating a given frozen set over the capacity-equivalent BEC instead of directly dealing with the BI-AWGN channel, which enables the selection of a favorable design SNR without time-consuming simulations.

The remainder of this paper is organized as follows. Section II describes preliminaries about polar codes and a channel model. Section III presents the proposed BEC-aided design SNR selection method. Section IV evaluates the proposed design through numerical simulations. Finally, Section V concludes the paper.

II. PRELIMINARIES

A. Polar Codes

We denote a binary input vector to a polar encoder of length N by $\mathbf{u} = (u_0, u_1, \dots, u_{N-1}) \in \mathbb{F}_2^N$ and the corresponding output codeword by $\mathbf{c} = (c_0, c_1, \dots, c_{N-1}) \in \mathbb{F}_2^N$, which is expressed as $\mathbf{c} = \mathbf{u} \mathbf{B}_N \mathbf{G}_N$, where $\mathbf{B}_N$ is the bit reversal permutation matrix. The generator matrix $\mathbf{G}_N$ is given by $\mathbf{G}_N = \mathbf{G}_2^{\otimes n}$, $\mathbf{G}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, where $n = \log_2 N$ and $\mathbf{G}_2^{\otimes n}$ is the n th Kronecker power of $\mathbf{G}_2$. This paper focuses on conventional polar codes with fixed zero-valued frozen bits and primitive SCL decoding, i.e., without cyclic redundancy check (CRC) or additional constraints such as dynamic frozen bits or convolutional precoding.

B. Design SNR and AWGN Construction

The polar code of rate $R = K/N$ is constructed by selecting the set of K most reliable bit channels in $\mathbf{u}$ that are used for information transmission, while the other bit channels are frozen. Let i denote the synthetic channel index and $\mathcal{Z} \triangleq \{0, 1, \dots, N-1\}$ denote the set of all indices. The information and frozen sets are denoted by $\mathcal{I} \subset \mathcal{Z}$ and $\mathcal{F} \triangleq \mathcal{Z} \setminus \mathcal{I}$, with $|\mathcal{I}| = K$ and $|\mathcal{F}| = N - K$.

For conventional SC-oriented construction methods such as GA and RCA, the reliability of each synthetic channel is estimated for a given design SNR γ^* . Let $\hat{r}_i(\gamma^*)$ denote the estimated reliability of the i -th synthetic channel by the conventional construction method such as GA or RCA. The information set is selected as a subset $\mathcal{I}(\gamma^*) \subset \mathcal{Z}$ with $|\mathcal{I}(\gamma^*)| = K$ satisfying

$$\hat{r}_i(\gamma^*) \geq \hat{r}_j(\gamma^*), \quad \forall i \in \mathcal{I}(\gamma^*), \quad \forall j \in \mathcal{Z} \setminus \mathcal{I}(\gamma^*). \quad (1)$$

The conventional choice is $\gamma^* = \gamma_{\text{ch}}$, where γ_{ch} is the operating SNR. In this paper, however, γ^* is treated as a design parameter for finite-list SCL decoding. Throughout this paper, we adopt RCA [4] as a conventional construction method for SC decoding.

C. SCL Decoding on the BEC

For a retained candidate path, let $\hat{u}_j$ denote the estimated value of u_j , and define $\hat{\mathbf{u}}_0^{i-1} = (\hat{u}_0, \dots, \hat{u}_{i-1})$. Given the received vector $\mathbf{y} = (y_0, \dots, y_{N-1})$ and the previous estimates $\hat{\mathbf{u}}_0^{i-1}$, the SC recursion for bit u_i on the BEC produces the tentative output $\tilde{u}_i(\mathbf{y}, \hat{\mathbf{u}}_0^{i-1}) \in \{0, 1, \text{e}\}$, where e denotes an

erasure. SCL decoding applies this recursion to each retained candidate path, with at most L paths retained.

If $i \in \mathcal{I}$, a non-erased tentative output $\tilde{u}_i \in \{0, 1\}$ extends the path only with its determined value, i.e., $\hat{u}_i = \tilde{u}_i$, whereas $\tilde{u}_i = \text{e}$ permits both values and causes a *path split*, generating two descendant paths with $\hat{u}_i = 0$ and $\hat{u}_i = 1$.

If $i \in \mathcal{F}$, we impose the frozen-bit constraint $u_i = 0$. Tentative outputs of $\tilde{u}_i \in \{0, \text{e}\}$ are consistent with the frozen-bit constraint and thus extend the path with $\hat{u}_i = 0$, whereas a tentative output of $\tilde{u}_i = 1$ violates the frozen-bit constraint and thus the corresponding candidate path is removed. We refer to the latter event as a *frozen-bit violation*.

Consequently, path splits can increase the number of candidate paths, whereas frozen-bit violations can decrease it. When the number of candidate paths exceeds L , *path pruning* retains L paths uniformly at random, since the BEC observations provide no basis for preferring one candidate path over another.

D. Channel Model

We consider binary phase shift keying (BPSK) transmission of a binary polar code over a BI-AWGN channel. Each coded bit $c_i \in \{0, 1\}$ is mapped to $x_i = 1 - 2c_i \in \{+1, -1\}$. Thus, the averaged symbol energy is normalized to $E_s = 1$. The received signal is expressed as

$$y_i = x_i + n_i, \quad n_i \sim \mathcal{N}(0, N_0/2). \quad (2)$$

In this paper, we define the channel SNR as $\gamma_{\text{ch}} = E_s/N_0 = 1/N_0$.

III. BEC-AIDED DESIGN SNR SELECTION

Our goal is to determine an appropriate design SNR for a prescribed list size with low computational complexity. To avoid time-consuming SCL decoding simulations over the BI-AWGN channel, an information set $\mathcal{I}(\gamma^*)$ is first constructed for the BI-AWGN channel through RCA, and its suitability for a given list size is then evaluated using a proposed metric derived from the analysis of SCL decoding over a capacity-equivalent BEC.

A. Capacity Equalization

Let $C_{\text{BI-AWGN}}(\gamma_{\text{ch}})$ denote the BPSK-constrained capacity of the BI-AWGN channel at operating SNR γ_{ch} , and let ϵ_{ch} denote the channel erasure probability of the BEC. We define the erasure probability of the capacity-equivalent BEC as

$$\epsilon_{\text{eq}} \triangleq 1 - C_{\text{BI-AWGN}}(\gamma_{\text{ch}}). \quad (3)$$

Upon developing the proposed method, we make use of the BEC with $\epsilon_{\text{ch}} = \epsilon_{\text{eq}}$.

B. Error-Event Decomposition

Let $\mathcal{A}_{\text{rem}}$ denote the event that the correct path is removed from the list during SCL decoding, and let $\mathcal{E}_{\text{dec}}$ denote the block decoding error event. Since correct decoding is infeasible once the correct path is removed, $\mathcal{A}_{\text{rem}} \subseteq \mathcal{E}_{\text{dec}}$. Hence, the BLER of SCL decoding is decomposed as

$$\Pr(\mathcal{E}_{\text{dec}}) = \Pr(\mathcal{A}_{\text{rem}}) + \Pr(\overline{\mathcal{A}_{\text{rem}}} \cap \mathcal{E}_{\text{dec}}). \quad (4)$$

The first term represents the probability of correct path removal due to the finite list size, whereas the second term represents a final path selection error when the correct path remains in the list. Since SCL decoding approaches ML decoding as the list size limitation becomes negligible, we characterize the latter error mechanism using an ML decoding-based measure.

C. Approximate List Saturation Probability

First, we note that exact evaluation of the path removal probability $\Pr(\mathcal{A}_{\text{rem}})$ may not be straightforward. The correct path can be removed only when path pruning occurs, and whether it survives each pruning operation is random. For simplicity, rather than modeling this random survival process, we focus on the list saturation event. Let $\mathcal{A}_{\text{sat}}$ denote the event that the number of candidate paths exceeds the list size L , resulting in a path-pruning operation. If $\mathcal{A}_{\text{sat}}$ does not occur, path pruning is never invoked and the correct path cannot be removed. Therefore, we obtain

$$\mathcal{A}_{\text{rem}} \subseteq \mathcal{A}_{\text{sat}}. \quad (5)$$

Nevertheless, evaluating $\Pr(\mathcal{A}_{\text{sat}})$ remains difficult since the number of candidate paths changes through path splits and frozen-bit violations. For a fixed received erasure pattern, the locations of path splits are common to all candidate paths. In contrast, the candidate paths that undergo a frozen-bit violation depend on their previously decoded bits. Hence, an exact evaluation of $\Pr(\mathcal{A}_{\text{sat}})$ requires tracking the set of candidate paths and the frozen-bit violations for each received erasure pattern.

To obtain a tractable upper-bound event for $\mathcal{A}_{\text{sat}}$, we introduce a *frozen-bit violation-free model*. In this model, candidate paths are not removed by frozen-bit violations, whereas path pruning is still performed. Thus, every erased information bit causes a path split and doubles the number of candidate paths, while a frozen-bit constraint does not reduce this number. Let $\mathcal{A}_{\text{vf}}$ denote the event that the number of candidate paths exceeds L before a path pruning under the frozen-bit violation-free model. Until the first path-pruning operation, this model retains every path that remains in the actual SCL decoder. Hence, whenever $\mathcal{A}_{\text{sat}}$ occurs, so does $\mathcal{A}_{\text{vf}}$, and therefore we obtain

$$\mathcal{A}_{\text{sat}} \subseteq \mathcal{A}_{\text{vf}}. \quad (6)$$

By combining (5) with (6), we obtain

$$\Pr(\mathcal{A}_{\text{rem}}) \leq \Pr(\mathcal{A}_{\text{sat}}) \leq \Pr(\mathcal{A}_{\text{vf}}). \quad (7)$$

Let E_i denote the random variable defined as the erasure indicator of the i -th synthetic channel, i.e., $E_i \triangleq \mathbf{1}\{\tilde{u}_i = e\}$, where $\mathbf{1}\{\cdot\}$ denotes the indicator function. Under the frozen-bit violation-free model, if path pruning were not applied, the number of candidate paths after processing all information bits in $\mathcal{I}$ would be 2^S , where $S \triangleq \sum_{i \in \mathcal{I}} E_i$. Since S is nondecreasing as decoding proceeds, the list saturation event occurs if and only if $2^S > L$. Therefore, given $a_L = \lfloor \log_2 L \rfloor$, $\mathcal{A}_{\text{vf}}$ is expressed as $\mathcal{A}_{\text{vf}} = \{S > a_L\}$. For the BEC with erasure

probability ϵ_{eq} , the erasure probability of the i -th synthetic channel is given by $\Pr(E_i = 1) = Z_i(\epsilon_{\text{eq}})$, where $Z_i(\epsilon_{\text{eq}})$ denotes the Bhattacharyya parameter of the i -th synthetic channel [1]. However, evaluating $\Pr(\mathcal{A}_{\text{vf}})$ requires the joint distribution of $\{E_i\}$ rather than their marginal probabilities as the erasure indicators are generally dependent. To deal with this difficulty, we introduce an independence approximation. Specifically, we replace each E_i with an independent Bernoulli random variable $\tilde{E}_i$ satisfying

$$\tilde{E}_i \sim \text{Bernoulli}(Z_i(\epsilon_{\text{eq}})), \quad i \in \mathcal{I}. \quad (8)$$

We then consider the approximate list saturation probability defined as an approximation to the frozen-bit violation-free saturation probability under the assumption that the erasure indicators are independent:

$$P_{\text{sat}}^{\text{app}}(\mathcal{I}, L) \triangleq \Pr\left(\sum_{i \in \mathcal{I}} \tilde{E}_i > a_L\right). \quad (9)$$

This probability can be evaluated by dynamic programming. Let $i_1 < i_2 < \dots < i_K$ denote the information bit indices in decoding order, and let us define the number of erasures among the first t information bits and the truncated erasure count as

$$\tilde{S}_t \triangleq \sum_{\tau=1}^t \tilde{E}_{i_\tau}, \quad D_t \triangleq \min\{\tilde{S}_t, a_L + 1\}, \quad (10)$$

respectively, where the case $D_t = a_L + 1$ represents an absorbing state with more than a_L erasures. For $t = 0, \dots, K$ and $s \in \{0, \dots, a_L + 1\}$, let us define the probability that the truncated erasure count equals s after processing the first t information bits as $B_s^{(t)} = \Pr(D_t = s)$. When processing the t -th information bit, the erasure count remains unchanged with probability $1 - Z_{i_t}(\epsilon_{\text{eq}})$ and increases by one with probability $Z_{i_t}(\epsilon_{\text{eq}})$, subject to truncation at $a_L + 1$. Therefore, we obtain $P_{\text{sat}}^{\text{app}}(\mathcal{I}, L) = B_{a_L+1}^{(K)}$. The above algorithm is summarized in **Algorithm 1**.

D. Proposed Selection Rule

For the candidate information set $\mathcal{I}$, we evaluate the MWUB given by

$$P_{\text{MWUB}}^{\text{BEC}}(\mathcal{I}) = \frac{1}{2} A_{d_{\min}}(\mathcal{I}) \epsilon_{\text{eq}}^{d_{\min}(\mathcal{I})}, \quad (11)$$

where $d_{\min}(\mathcal{I})$ and $A_{d_{\min}}(\mathcal{I})$ denote the minimum distance and the number of minimum-weight codewords, respectively. The closed-form expression of $A_{d_{\min}}(\mathcal{I})$ for the polar code defined by $\mathcal{I}$ is presented in [20]. Since the BLER under ML decoding over the BEC is dominated by minimum-weight codewords in the low-erasure-probability regime, we use the MWUB as a measure for the distance-limited ML decoding behavior.

The BLER of SCL decoder is no smaller than both the path removal probability and the BLER under ML decoding. Motivated by this relation, the proposed BEC-domain design metric is defined as the larger of the two measures as

$$J_{\text{BEC}}(\mathcal{I}, L) \triangleq \max\{P_{\text{MWUB}}^{\text{BEC}}(\mathcal{I}), P_{\text{sat}}^{\text{app}}(\mathcal{I}, L)\}. \quad (12)$$

Algorithm 1 Computation of the approximate list saturation probability

Input: $N, K, L, \mathcal{I}(\gamma^*), \{Z_i(\epsilon_{\text{eq}})\}_{i=0}^{N-1}$

Output: $P_{\text{sat}}^{\text{app}}$

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1: Sort the information set in decoding order as  $\mathcal{I}(\gamma^*) = \{i_1, i_2, \dots, i_K\}$ , where  $i_1 < i_2 < \dots < i_K$ 
2:  $a_L \leftarrow \lfloor \log_2 L \rfloor$ 
3:  $B_0 \leftarrow 1$  and  $B_s \leftarrow 0$  for  $s = 1, \dots, a_L + 1$ 
4: for  $t = 1, 2, \dots, K$  do
5:   Set  $B'_s \leftarrow 0$  for  $s = 0, \dots, a_L + 1$ 
6:   for  $s = 0, 1, \dots, a_L + 1$  do
7:      $B'_s \leftarrow B'_s + B_s(1 - Z_{i_t}(\epsilon_{\text{eq}}))$ 
8:      $s^+ \leftarrow \min\{s + 1, a_L + 1\}$ 
9:      $B'_{s^+} \leftarrow B'_{s^+} + B_s Z_{i_t}(\epsilon_{\text{eq}})$ 
10:  end for
11:   $B_s \leftarrow B'_s$  for all  $s = 0, \dots, a_L + 1$ 
12: end for
13:  $P_{\text{sat}}^{\text{app}} \leftarrow B_{a_L+1}$ 

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Algorithm 2 Proposed BEC-aided design SNR selection

Input: $N, K, L, \gamma_{\text{ch}}, \Delta\gamma^*, M$

Output: γ_{prop}^*

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1:  $J_{\min} \leftarrow \infty$ 
2:  $\epsilon_{\text{eq}} \leftarrow 1 - C_{\text{BI-AWGN}}(\gamma_{\text{ch}})$ 
3: Compute  $\{Z_i(\epsilon_{\text{eq}})\}_{i=0}^{N-1}$  by [1]
4: for  $m = 1, 2, \dots, M$  do
5:    $\gamma^* \leftarrow \gamma_{\text{ch}} + (m - 1)\Delta\gamma^*$ 
6:    $\mathcal{I}(\gamma^*) \leftarrow$  information set constructed at  $\gamma^*$ 
7:   Compute  $d_{\min}$  and  $A_{d_{\min}}$  by [20]
8:   Compute  $P_1 \leftarrow P_{\text{MWUB}}^{\text{BEC}}(\mathcal{I}(\gamma^*))$  by (11)
9:   Compute  $P_2 \leftarrow P_{\text{sat}}^{\text{app}}(\mathcal{I}(\gamma^*), L)$  by Algorithm 1
10:   $J_{\text{BEC}} \leftarrow \max\{P_1, P_2\}$ 
11:  if  $J_{\text{BEC}} \leq J_{\min}$  then
12:     $J_{\min} \leftarrow J_{\text{BEC}}$ 
13:     $\gamma_{\text{prop}}^* \leftarrow \gamma^*$ 
14:  else
15:    break
16:  end if
17: end for

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For the design SNR search, we consider the uniformly spaced candidate set $\mathcal{G}^* = \{\gamma_{\text{ch}} + (m - 1)\Delta\gamma^* : m = 1, \dots, M\}$, where $\Delta\gamma^*$ denotes the design SNR step size and $M = |\mathcal{G}^*|$. The proposed BI-AWGN design SNR is obtained as

$$\gamma_{\text{prop}}^* = \arg \min_{\gamma^* \in \mathcal{G}^*} J_{\text{BEC}}(\mathcal{I}(\gamma^*), L). \quad (13)$$

In the implementation, the candidates are examined in increasing order of γ^* . Since J_{BEC} empirically exhibits a single trough over the considered design SNR range possibly with flat regions caused by unchanged information sets, the search is terminated when the metric first exceeds its best value obtained so far. This early-stopping rule is used only to reduce the number of candidate evaluations. The procedure is summarized in **Algorithm 2**.

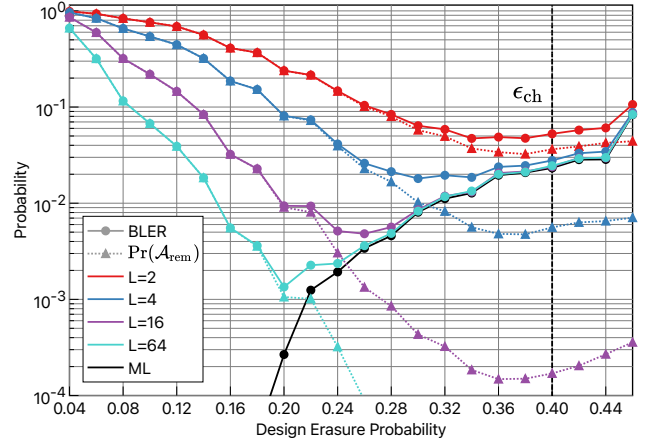


Fig. 1. BLERs of BEC-designed polar codes for $N = 1024$, $K = 512$, and $\epsilon_{\text{ch}} = 0.4$, together with the ML-decoding BLER and path removal probability $\Pr(\mathcal{A}_{\text{rem}})$.

E. Computational Complexity

For each candidate design SNR, the proposed method requires RCA-based construction with complexity $O(N \log N)$, closed-form evaluation of the MWUB term with complexity $O(K \log N)$, and the calculation of $P_{\text{sat}}^{\text{app}}$ with complexity $O(K \log L)$. Since no Monte Carlo SCL decoding simulations are required during the search, the complexity is substantially lower than that of exhaustive BLER-based optimization.

IV. SIMULATION RESULTS

This section evaluates the proposed BEC-aided design SNR selection method through numerical simulations. We first examine the error-event decomposition by comparing correct path removal probability and ML-decoding BLER over the BEC, and then validate the two measures used in the proposed metric. We next compare the design SNR selected by the proposed method with that obtained by exhaustive search. Finally, we compare the resulting BLER performance with existing constructions over a channel SNR range of interest.

A. Error-Event Decomposition Evaluation

Fig. 1 shows the simulated BLER of BEC-designed polar codes for $N = 1024$, $K = 512$, and $\epsilon_{\text{ch}} = 0.4$ as a function of the design erasure probability, together with the ML-decoding BLER and the correct path removal probability $\Pr(\mathcal{A}_{\text{rem}})$ obtained by simulation. The polar codes are constructed by selecting the K bit channels with the smallest Bhattacharyya parameters as information channels. For moderate and large list sizes, the figure suggests that a favorable design erasure probability is located near the transition between the list-limited regime, represented by $\Pr(\mathcal{A}_{\text{rem}})$, and the distance-limited regime, represented by the ML curve. For $L = 2$, however, correct path removal event dominates over the considered range, and the gain from changing the design erasure probability is therefore limited. These results support the interpretation of the two error mechanisms as finite-list behavior associated with correct path removal and distance-limited behavior associated with ML decoding.

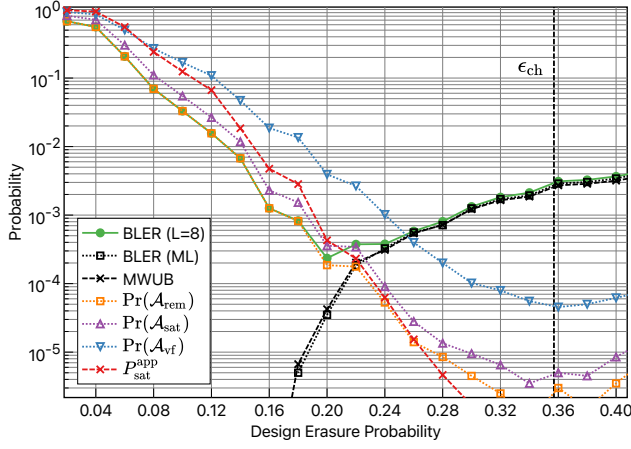


Fig. 2. BEC-aided metric evaluation for $N = 1024$, $K = 512$, and $L = 8$ as functions of the BEC design erasure probability. The vertical dashed line indicates $\epsilon_{ch} = 0.357$.

B. Validation of the Proposed Metric

To examine the behavior of the proposed metric, we evaluate its component terms on the BEC itself. Fig. 2 compares $\Pr(\mathcal{A}_{rem})$, $\Pr(\mathcal{A}_{sat})$, $\Pr(\mathcal{A}_{vf})$, and P_{sat}^{app} . The ML decoding BLER and P_{MWUB}^{BEC} are also shown. $\Pr(\mathcal{A}_{rem})$, $\Pr(\mathcal{A}_{sat})$, $\Pr(\mathcal{A}_{vf})$, and ML decoding BLER are obtained by computer simulation, while P_{sat}^{app} and P_{MWUB}^{BEC} are given by (9) and (11), respectively. The BEC erasure probability is $\epsilon_{ch} = \epsilon_{eq} = 0.357$, which is capacity-equivalent to the BI-AWGN channel at $\gamma_{ch} = -1$ dB. We focus on $L = 8$. Fig. 2 shows that P_{sat}^{app} captures the variation of the finite-list limitation with the design erasure probability. Together with the MWUB term, it identifies a crossover region close to the design erasure probability that minimizes the BLER.

C. Comparison between Exhaustive-Search and Proposed Design SNRs

Fig. 3 and Fig. 4 show the BLER as a function of the design SNR at the fixed operating SNRs for $R = 1/2$ and $3/4$. For the proposed method, the candidate design SNRs are examined with a resolution of $\Delta\gamma^* = 0.1$ dB. The vertical lines mark the channel SNR (dashed line) and the proposed BEC-aided design SNRs (dotted lines). These figures show that the proposed design SNR is located near the bottom region of the BLER curve, whereas the channel-SNR design can be far from the best design point, especially for larger list sizes.

Table I compares, in dB, the proposed design SNR γ_{prop}^* and the design SNR γ_{exh}^* selected by exhaustive BLER-based search. For exhaustive search, the candidate information sets are evaluated by Monte Carlo simulations while sweeping the design SNR with a resolution of $\Delta\gamma^* = 0.1$ dB. The design SNR prediction error is defined as $\Delta\gamma_{pred}^* \triangleq |\gamma_{prop}^* - \gamma_{exh}^*|$. For notational simplicity, let $P_{BL}(\gamma^*)$ denote the simulated BLER at the operating SNR γ_{ch} for a code constructed with design SNR γ^* . We quantify the logarithmic BLER gains of exhaustive search and the proposed method, relative to the conventional channel-SNR design, as

$$G_{exh} \triangleq \log_{10} \frac{P_{BL}(\gamma_{ch})}{P_{BL}(\gamma_{exh}^*)}, \quad G_{prop} \triangleq \log_{10} \frac{P_{BL}(\gamma_{ch})}{P_{BL}(\gamma_{prop}^*)}. \quad (14)$$

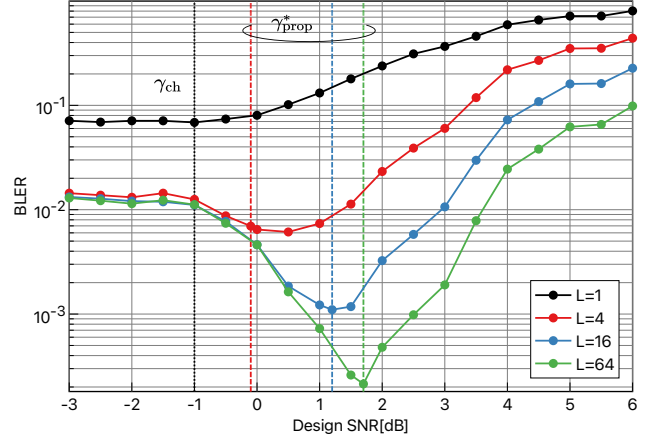


Fig. 3. BLER as a function of the design SNR over the BI-AWGN channel for $N = 1024$, $K = 512$, and $R = 1/2$. The vertical dashed line indicates the operating SNR $\gamma_{ch} = -1.0$ dB, and the dotted lines indicate the proposed BEC-aided design SNRs γ_{prop}^* for the respective list sizes.

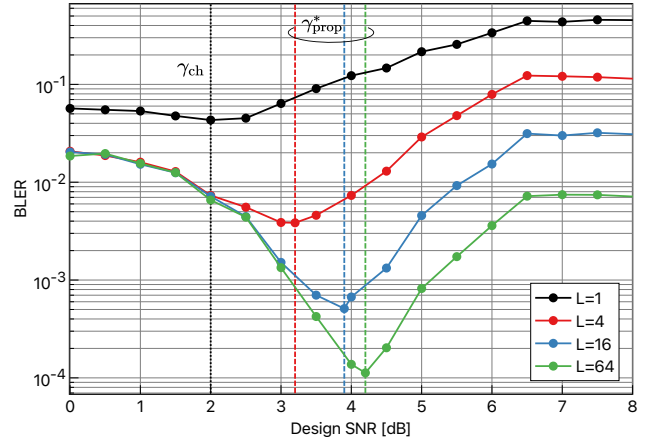


Fig. 4. BLER as a function of the design SNR over the BI-AWGN channel for $N = 1024$, $K = 768$, and $R = 3/4$. The vertical dashed line indicates the operating SNR $\gamma_{ch} = 2.0$ dB, and the dotted lines indicate the proposed BEC-aided design SNRs γ_{prop}^* for the respective list sizes.

The fraction of the exhaustive-search gain recovered by the proposed method is then defined as $\eta \triangleq G_{prop}/G_{exh}$, where $\eta = 1$ indicates that the proposed method attains the same BLER as exhaustive search.

As shown in Table I, the proposed design SNR is close to that selected by the exhaustive BLER-based search for moderate and large list sizes. The attainable gain from exhaustive design SNR optimization, quantified by G_{exh} , is small for $L = 2$ but increases as the list size becomes larger. For moderate and large list sizes, G_{prop} is close to G_{exh} and the corresponding recovery ratio η is close to one in most cases. These results indicate that, when design SNR optimization provides a meaningful BLER improvement, the proposed BEC-aided method recovers most of the available BLER gain.

D. BLER Performance over the BI-AWGN Channel

Fig. 5 compares the BLER performance of the RCA construction designed at the channel SNR, the proposed RCA-based construction, and the BLER-based search construction

TABLE I
COMPARISON BETWEEN EXHAUSTIVE-SEARCH DESIGN SNR AND
PROPOSED BEC-AIDED DESIGN SNR (dB).

R	γ_{ch}	L	γ_{exh}^*	γ_{prop}^*	$\Delta\gamma_{\text{pred}}^*$	G_{exh}	G_{prop}	η
1/2	-1.0	2	0.0	-1.0	1.0	0.10	0.00	0.00
		4	0.4	-0.1	0.5	0.35	0.26	0.74
		8	1.0	0.9	0.1	0.68	0.65	0.96
		16	1.2	1.2	0.0	1.00	1.00	1.00
		32	1.7	1.3	0.4	1.35	1.28	0.95
3/4	2.0	64	1.7	1.7	0.0	1.66	1.66	1.00
		2	2.6	2.5	0.1	0.10	0.05	0.53
		4	3.3	3.2	0.1	0.30	0.28	0.92
		8	3.9	3.5	0.4	0.73	0.70	0.96
		16	3.9	3.9	0.0	1.15	1.15	1.00
		32	4.0	4.2	0.2	1.45	1.32	0.92
		64	4.2	4.2	0.0	1.77	1.77	1.00

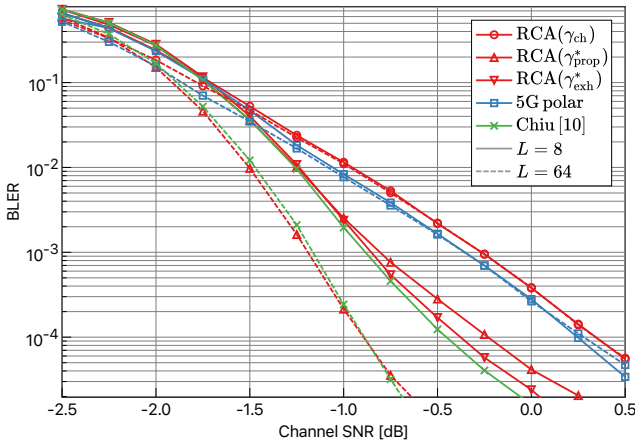


Fig. 5. BLER comparison over the BI-AWGN channel for $N = 1024$ and $R = 1/2$. Solid and dashed curves correspond to $L = 8$ and 64 , respectively. Note that $\gamma_{\text{prop}}^* = \gamma_{\text{exh}}^*$ for $L = 64$, and thus the corresponding BLER curves overlap.

designed for the target operating SNR $\gamma_{\text{ch}} = -1.0$ dB, together with the 5G polar code and the construction based on Chiu [10]. We consider $N = 1024$, $R = 1/2$, and list sizes $L = 8$ and 64 . For the construction in [10], BLER_T denotes the target BLER, and P denotes the number of candidate information bits considered at each selection step. We set $\text{BLER}_T = 10^{-5}$, and P is selected by Monte Carlo simulations from the considered candidates, yielding $P = 12$ for $L = 8$ and $P = 30$ for $L = 64$.

For $L = 8$, the proposed method selects $\gamma_{\text{prop}}^* = 0.9$ dB, whereas the BLER-based search selects $\gamma_{\text{exh}}^* = 1.0$ dB. Their BLER curves are nearly identical at the channel SNR $\gamma_{\text{ch}} = -1.0$ dB. For $L = 64$, both methods select $\gamma_{\text{prop}}^* = \gamma_{\text{exh}}^* = 1.7$ dB among the considered candidate design SNRs. Furthermore, in both cases, the proposed construction shows better BLER performance than the RCA construction designed at the channel SNR and the 5G polar code, while providing BLER performance competitive with that of the construction in [10].

V. CONCLUSION

This paper studied design SNR optimization of polar codes for finite-list SCL decoding. We proposed a BEC-aided low-complexity design SNR selection method. The proposed

method evaluates the candidate information sets constructed for BI-AWGN over a capacity-equivalent BEC by combining an MWUB term and an approximate list saturation probability. Numerical results showed that the proposed method is able to identify design SNRs whose BLER performance is close to that obtained by exhaustive search for moderate and large list sizes, without resorting to time-consuming Monte Carlo simulations.

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